Minimum Damping Needed for Vanishing an Unstable Pocket of a Hill Equation

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Summary. For the stability chart of a Hill equation we introduce the definition of Maximum Energy Lines (MEL), these lines are composed of points where the unstable solutions of the differential equation reaches its maximum growth. Then we use the definition of MEL in order to give a new characterization of the coexistence points and to find the minimum damping needed to vanish an unstable pocket, if they exist. Analytical and numerical results, for Meissner equation are presented.

Introduction

An homogeneous, linear, second-order differential equations with real, periodic coefficients of the form $\ddot{x}+(\alpha+\beta q(t))\,x=0$ with $q(t+T)=q(t)$ is called Hill's equation [3]. This kind of equations describes a vast number of periodic systems, from the movement of celestial bodies to the L-C electric circuits, maybe the most known and didactic example is the swinging of a mass attached to a periodic moving support, where the very unusual behaviour of the systems described by a Hill equation can be appreciated. Namely, if a force varying periodically acts on a mass in such manner that the force tends to move the mass back into its equilibrium point, one may expect that the mass stays within a neighbourhood of its equilibrium. Once the force is strong enough to achieve this effect, one would expect that a stronger force be more efficient, but, this may not be the case, the mass may oscillate wider and wider. Then, the stability of the solutions of a periodic differential equation may change with the slight change of parameters such as the amplitude and frequency of the periodic excitation.

The stability of solutions of a Hill's equation could be represented as the so called Ince-Strutt diagram, which consists on stable and unstable regions divided by transition curves in the plane of parameters $\alpha-\beta$. If $T=2\pi$ then, each unstable zone rises at the point $\left(\frac{n^2}{4},0\right)$ and they are called as the integer number which define where they rise. In Fig. 1a), the unstable regions are numbered as described above. Transition curves are characterized by having at least one T-periodic or $2T$- periodic solution [4]. Unstable regions are also called Arnold tongues. For a comprehensive survey on the subject see [7].

It is well known that transition curves of a Hill's equation having different period do not intersect each other at any point [3]. But, if two lines, with the same period, intersect each other then, there are two linear independent solutions of the Hill's equation with the same period and we shall call this "coexistence". For a detailed account of coexistence in Hill's equation see [8] and [9]. Along this paper, we refer as instability pockets to the unstable regions between two coexistence points of the same Arnold tongue.

The aim of this article is to give a method to vanish any instability pocket of any Hill equation by adding a dissipative term $\delta\dot{x}$ to the periodic differential equation and to give the minimum amount of dissipation δ needed to achieve this task. In section 2 we give some basic concepts about the Floquet theory, a characterization of the solutions of a linear periodic system and an useful theorem which relates Iso-μ curves with transition curves of a related damped Hill's equation, that is, the existing relationship between curves, in the $\alpha-\beta$ plane, where unstable solutions grow with the same velocity and the curves where there is at least one periodic solution of a related damped Hill's equation. In section 3 we introduce the definition of maximum energy lines (MELs) and the minimum damping needed to vanish any unstable pocket is obtained. In section 4 we found an analytical approximation of the MELs of Meissner equation and verify the main theorem given in section 3. Finally, some concluding remarks are given in section 5.

Preliminaries

In this section we collect some basic properties of periodic differential equations and present a theorem on the relation between the growth of the solutions and the solutions of a damped periodic system. The theorem proofs are omitted, but, we indicate where they can be found.

Consider the homogeneous linear periodic system

$$\dot{x}(t)=A(t)\,x(t) \tag{1}$$

where $A(t+T)=A(t)$ and $A(t)\in\mathbb{R}^{n\times n}$. A second order differential equation of the form

$$\ddot{y}+(\alpha+\beta q(t))\,y=0 \tag{2}$$

with $q(t+T)=q(t)$, and $y,\alpha,\beta\in\mathbb{R}$ is an example of a dynamic system which can be represented by (1). Equation (2) is known as Hill's equation. Next theorem, due to Floquet, states that the solutions of (1) may be factorized by three matrices, two periodic and one exponential, as follows

Theorem 1 *The state transition matrix $\Phi(t,t_0)\in\mathbb{R}^{n\times n}$ of (1) has the form*

$$\Phi(t,t_0)=P^{-1}(t)\,e^{B(t-t_0)}P(t_0)$$

where $P(t)$, B are $n\times n$ matrices, $P(t+T)=P(t)$ $\forall t$, B is constant, not necessarily real.

Proof 2 *The proof of Theorem 1 can be found in [10].* ■

Remark 3 *Notice that if we set $t = t_0$ then, $\Phi(t_0, t_0) = I_n$. The importance of the state transition matrix $\Phi(t, t_0)$ is that it maps any initial state $x(t_0)$ to a unique final state $x(t)$, namely, $x(t) = \Phi(t, t_0) x(t_0)$. Without loss of generality, if we set $t_0 = 0$ then, by Theorem 1, the state transition matrix can be written as*

$$\Phi(t, 0) = P^{-1}(t) e^{Bt}$$

The monodromy matrix of (1) is a non singular matrix M associated with the matrix $\Phi(t, t_0)$ through the relation $\Phi(t + T, t_0) = \Phi(t, t_0) M$ [11], in other words, the monodromy matrix is equal to the state transition matrix valuated at one period T,

$$M = \Phi(t_0 + T, t_0) \tag{3}$$

The eigenvalues μ_i, $i = 1, \ldots, n$, of the matrix M are called characteristic multipliers and any ρ_i such that $\mu_i = e^{\rho_i T}$ are called characteristic exponents. The exponents ρ_i could always be chosen as the eigenvalues of B where B is any matrix so that $M = e^{BT}$ [10].
Matrix M plays a fundamental role for the stability of the system (1). Let $t \geq 0$, if we write t as: $t = kT + \tau$, $\tau \in [0, \tau]$ and $t_0 = 0$, applying properties of the state transition matrix $\Phi(t, t_0)$ [11], one can write:

$$\begin{aligned} x(t) &= \Phi(t, 0) x_0 \\ &= \Phi(kT + \tau, 0) x_0 \\ &= \Phi(kT + \tau, kT) \Phi(kT, (k-1)T) \ldots \Phi(T, 0) x_0 \\ &= \Phi(\tau, 0) M^k x_0 \end{aligned} \tag{4}$$

from latter equation we can notice that since, x_0 and $\Phi(\tau, 0)$ are bounded, the stability of the solution of (1) depends on the matrix M and therefore on the characteristic multipliers μ_i. Next theorem gives the conditions for stability of the solutions of (1)

Theorem 4 *Let μ_i be the characteristic multipliers of (1), then:*
a) The solutions of (1) are asymptotically stable if and only if all $|\mu_i| < 1$.
b) The solutions of (1) are stable if and only if all $|\mu_i| \leq 1$, and if any μ_i has modulo one, it must be a simple root of the minimal polynomial of M.
c) The solutions of (1) are unstable if and only if there is a μ_i such that $|\mu_i| > 1$ or if all $|\mu_i| \leqslant 1$ and there is one $\mu_j : |\mu_j| = 1$ and μ_j is a multiple root of the minimal polynomial of M.

Theorem 1 is known as the Floquet theorem. Theorem 4 allows us to analyse the stability of (1) by knowing the characteristic multipliers μ_i, that is, by knowing the solution $\Phi(t, 0)$ at $t = T$.
Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the Hill equation (2), fulfilling the initial conditions

$$\begin{aligned} y_1(0) = 1 \qquad \dot{y}_1(0) = 0 \\ y_2(0) = 0 \qquad \dot{y}_2(0) = 1 \end{aligned}$$

and define the function

$$\Delta(\alpha, \beta) \triangleq y_1(T) + \dot{y}_2(T) \tag{5}$$

where T is the period of $q(t)$ in (2), then, the characteristic multipliers of equation (2) are defined by the characteristic polynomial

$$\det(\mu I_2 - M) = \mu^2 - \Delta(\alpha, \beta)\mu + 1 = 0$$

the linear term is equal 1 because of Liouville Theorem [1]. The function $\Delta(\alpha, \beta)$ is known as the discriminant of Hill equation. Solving the characteristic polynomial for μ we get

$$\mu_{1,2} = \frac{\Delta(\alpha, \beta) \pm \sqrt{\Delta(\alpha, \beta)^2 - 4}}{2}$$

so, the characteristic multipliers μ_i depends on $\Delta(\alpha, \beta)$ and therefore on the current value of the parameters α and β. Following the Theorem 4 one can say that the solutions of the Hill equation (2) are : a) Stable, If $|\Delta(\alpha, \beta)| < 2$, since, the multipliers are complex conjugated numbers and $|\mu_i| = 1$; b) Unstable, if $|\Delta(\alpha, \beta)| > 2$, since, both multipliers are real, one of them is $|\mu| < 1$ and the other is $|\mu| > 1$; c) There exist at least one periodic solution, if $|\Delta(\alpha, \beta)| = 2$, since both multipliers are $|\mu| = 1$, and if the Monodromy matrix is similar to a diagonal matrix, then this point is stable and corresponds to a coexistence point.

From the above mentioned and Theorem 4 one can infer that the stability of the solutions of the Hill equation (2) depend on the parameters α and β. Moreover, the plane of parameters $\alpha - \beta$ is splited into stable and unstable zones, the boundaries between the stable and unstable areas are called transition curves[1], see Fig. 1a).
It is known that the function $\Delta(\alpha, \beta)$ is an entire function and its order of growth is $\frac{1}{2}$ [2], this implies that the functions $\Delta(\alpha, \beta) + 2$ and $\Delta(\alpha, \beta) - 2$ have infinitely many zeros [3]. The following theorem due to Haupt gives us the relation between the function $\Delta(\alpha, \beta) = \pm 2$ and the stable and unstable zones, of a Hill's equation, in the plane of parameters $\alpha - \beta$.

Theorem 5 *To every differential equation (2) with β fixed, there belong two monotonically increasing infinite sequences of real numbers*

$$\alpha_0 < \alpha_1 \leq \alpha_2 < \alpha_3 \leq \dots \tag{6}$$
$$\alpha_1^* < \alpha_2^* \leq \alpha_3^* < \alpha_4^* \leq \dots \tag{7}$$

(6) is solution of $\Delta(\alpha, \beta) - 2 = 0$ and (7) is solution of $\Delta(\alpha, \beta) + 2 = 0$. The α_n and α_n^* satisfy the inequalities

$$\alpha_0 < \alpha_1^* \leq \alpha_2^* < \alpha_1 \leq \alpha_2 < \alpha_3^* \leq \dots$$

The solution of (2) is stable if α lies in the intervals

$$(\alpha_0, \alpha_1^*), (\alpha_2^*, \alpha_1), (\alpha_2, \alpha_3^*), (\alpha_4^*, \alpha_3), \dots$$

and the solution is unstable if α lies in the intervals

$$(-\infty, \alpha_0), (\alpha_1^*, \alpha_2^*), (\alpha_1, \alpha_2), (\alpha_3^*, \alpha_4^*), \dots \tag{8}$$

Proof 6 *The proof of Theorem 5 can be found in [3].* ■

Each unstable zone, in the $\alpha - \beta$ plane, is identified by an integer number $n = 0, 1, 2, \dots$, the assignment of this integer number depends on the position the unstable zone is; the zeroth unstable one is the one more to the left on the $\alpha - \beta$ plane, the next one to the right is the first, and so on, see Fig. 1a).

Iso-μ curves
Remember that the eigenvalues of the Monodromy matrix, are denoted as μ_i and called multipliers, then for some $|\mu_i| > 1$, the Iso-μ curves are lines inside the regions of instability. In particular the Iso-μ curves of a Hill's equation are defined by values of α and β for which the solutions have the same growth rate[2], see [5], [12]. As a consequence of Floquet theorem the solution of any periodic differential equation could be written as

$$x(t) = e^{\rho_i t} p(t)$$

where $p(t+T) = p(t)$ [10], so the solution will be unstable if the real part of the characteristic exponent ρ_i is a positive number, so there exists a relationship between the growth rate and the characteristic exponents ρ_i.
The growth rate is proportional to the real part of the characteristic exponents ρ_i. The condition for the solution of (1) grows exponentially is $|\mu_i| > 1$ or equivalently $\mathrm{Re}(\rho_i) > 0$. Then, there also exists a relationship between the growth rate and the characteristic multiplier μ_i. We will define the maximum growth rate as

$$\gamma = \max\{|\mu_i| : \mu_i \in \sigma(M)\} > 1$$

where $\sigma(M) = \{\mu_1, \mu_2, \dots, \mu_n\}$.
Therefore the Iso-μ curves are lines in the unstable zones, on the $\alpha - \beta$ plane, where the solution of a Hill's equation have the same growth rate γ. Fig. 1b) shows some Iso-μ curves of the Meissner equation, $\ddot{x} + (\alpha + \beta sign(\cos(t)))\, x = 0$, for different values of γ.
The following theorem gives us the existing relation between the Iso-μ curves and the transition curves of a different but related damped Hill's equation [6].

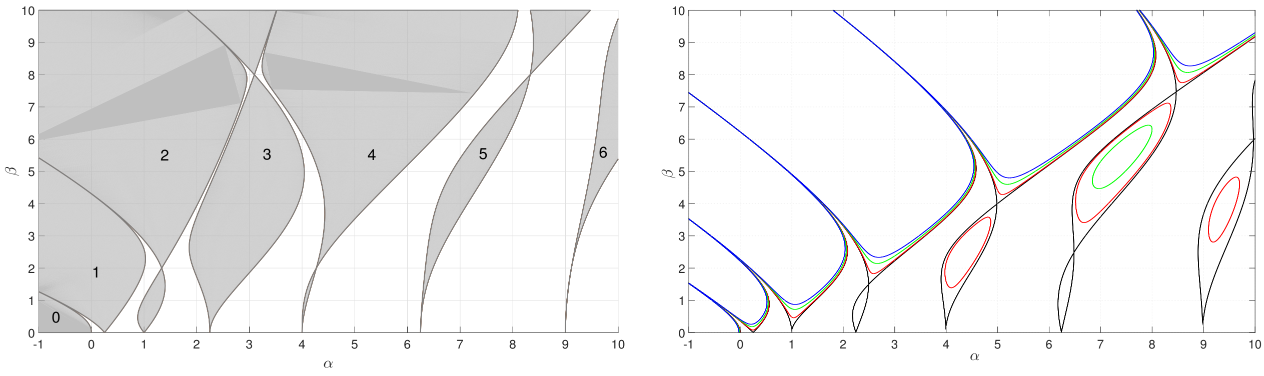
Theorem 7 *An Iso-μ curve of a Hill's equation*

$$\ddot{y} + (\alpha + \beta q(t))\, y = 0 \tag{9}$$

where $\alpha, \beta \in \mathbb{R}$, $y \in \mathbb{R}$ and $q(t+T) = q(t)$, with some growth rate γ, is equal to the transition curve of a related damped Hill's equation

$$\ddot{x} + \delta\dot{x} + (\alpha^* + \beta q(t))\, x = 0 \tag{10}$$

[1]Transition curves are composed by points, in the $\alpha - \beta$ plane, for which there is at least one periodic solution of the associated periodic differential equation, that is for α, β values such that $|\Delta(\alpha, \beta)| = \pm 2$.
[2]With growth rate we mean how fast a solution goes to infinity.



(a) (b)

Figure 1: a) Stable zones in white, unstable zones in gray and transition curves in black continuous lines for the equation $\ddot{x} + (\alpha + \beta(cos(t) + cos(2t)))x = 0$. b) Iso-$\mu$ curves for different values of γ: $\gamma = 1.3691$ in red, $\gamma = 2.0536$ in green and $\gamma = 2.7382$ in blue. Transition curves in black. For Meissner equation $\ddot{x} + (\alpha + \beta sign(cos(t)))x = 0$

where $\alpha^*, \delta \in \mathbb{R}$, $x \in \mathbb{R}$, if and only if

$$\gamma = e^{\frac{1}{2}\delta T} \tag{11}$$

and

$$\alpha = \alpha^* - \frac{1}{4}\delta^2$$

where $\gamma = \max\left[\left|\sigma\left(\Phi_y(T, 0)\right)\right|\right]$.

Proof 8 *The proof of Theorem 7 can be found in [6].*■

The interesting part of this Theorem is that it establishes a relation between the original Hill's equation (9) and some related one (10) in the sense that the transition curves of (10) coincide with the Iso-μ curves of (9); moreover it gives us an easy way to calculate the Iso-μ curves of (9).

MEL and the minimum damping needed to vanish an unstable pocket

Theorem 5 implies that, for each fixed β, the transition curves of a Hill's equation are defined by the zeros of the functions $\Delta(\alpha, \beta) - 2$ and $\Delta(\alpha, \beta) + 2$. From a theorem due to Laguerre [13] which says that *if $f(z)$ is an entire function, real for real z, of growth order less than 2, with real zeros, then the zeros of $\frac{d}{d_z} f(z)$ are also all real, and are separated from each other by the zeros of $f(z)$.* One can say that, for a fixed β, only one zero of $\frac{\partial}{\partial\alpha}\Delta(\alpha, \beta)$ will be inside of each unstable interval (8). And we came up to the following definitions.

Definition 9 *For β fixed, the maximum energy point $\phi_n(\beta)$, in each unstable interval (8), is the value of α for which the modulo of the maximum characteristic multiplier reach a maximum value, and depends on the current value of β. The sub index n refers to the appearance order, being the first the smallest one.*

Definition 10 *The maximum energy line (MEL) is the union of all the maximum energy points $\phi_n(\beta)$ with the same sub index n, that is*

$$MEL_n = \cup_{\forall\beta}\phi_n(\beta)$$

we will represent the MEL of the nth unstable zone as MEL_n.

Remark 11 *In the particular case of (2) the maximum energy points $\phi_n(\beta)$ are defined as the (α, β) values where $\frac{\partial}{\partial\alpha}\Delta(\alpha, \beta) = 0$ and β fixed. And the MELs are continuous lines depending on β and the number of unstable region where the line lays. Fig. 2 shows the MELs for the Meissner equation.*

As the MEL_n lays inside the nth unstable regions, then the MEL_n rises from the same α the unstable region rises, i. e. if $T = 2\pi$ then, $\phi_n(0) = \frac{n^2}{4}$. And, if the nth unstable region vanish at some β, i. e. there exists a coexistence point in the unstable region, then the MEL_n goes through the coexistence point. Next theorem gives us an analytic characterization of the coexistence points.

Theorem 12 *An $(\alpha_0, \beta_0) \in MEL_{2n}$ is a T-periodic coexistence point if and only if $\Delta(\alpha_0, \beta_0) = 2$. And, an $(\alpha_0, \beta_0) \in MEL_{2n+1}$ is a $2T$-periodic coexistence point if and only if $\Delta(\alpha_0, \beta_0) = -2$.*

Remark 13 *One can prove that each MEL_n fulfil the following properties:*
1. *The MEL_n rises from the same α as the nth Arnold tongue (unstable region) [3], i.e. if $T = 2\pi$ then, $\phi_n(0) = \frac{n^2}{4}$.*
2. *The MEL_n goes through the coexistence points of the nth Arnold tongue, if they exist.*
3. *If there exist m coexistence points along the MEL_n then $\frac{\partial}{\partial\beta}\Delta(\phi_n(\beta), \beta) = 0$ for m different values of β; i.e. the discriminant along the MEL_n will have m critical values.*
4. *If there do not exist coexistence points along the MEL_n then $|\Delta(\phi_n(\beta), \beta)| \to \infty$ as $\beta \to \infty$, and $\Delta(\phi_n(0), 0) = 2$.*

Minimum damping needed to vanish an unstable pocket

From Fig. 1b) one can notice that the maximum energy of an unstable pocket is located near its center, and the region with less energy are near the coexistence points and transition curves. We know that $|\Delta(\alpha, \beta)| = 2$ if and only if α and β are over the transition curves. So if a MEL goes through a coexistence point, then, $|\Delta(\phi_n(\beta), \beta)| = 2$ for more than one value β, which means that $\Delta(\phi_n(\beta), \beta)$ will has some critical values. The number of this critical values will depend only on the number of coexistence points. The following theorem gives us the condition to vanish an unstable pocket[3].

Lemma 14 *Let $(\bar{\alpha}, \bar{\beta})$ be the point of maximum energy inside an unstable pocket of a Hill equation and $\psi \triangleq \Delta(\bar{\alpha}, \bar{\beta})$. The unstable pocket will vanish if a damping term $\delta\dot{y}$ is added to (2) and the minimum amount of damping δ needed to vanishing it is*

$$\delta = \frac{2}{T}\ln(\mu) \tag{12}$$

where T is the minimal period of $q(t)$ in (2) and μ is the maxim modulo of the characteristic multipliers at $(\bar{\alpha}, \bar{\beta})$, i. e. $\mu = \max\left\{\left|\frac{\psi \pm \sqrt{\psi^2 - 4}}{2}\right|\right\}$.

Proof 15 *Let $\Delta(\alpha, \beta)$ be the discriminant of a Hill's equation (2). In order to obtain the maximum energy points and the MELs, one has to calculate the partial derivative with respect to α and equating to zero.*

$$\frac{\partial}{\partial\alpha}\Delta(\alpha, \beta) = 0 \tag{13}$$

Suppose that $\phi_n(\beta)$ represents the solution of (13), i.e. the maximum energy points, of the n unstable region having at least one coexistence point. Substituting $\phi_n(\beta)$ instead of α in $\Delta(\alpha, \beta)$ we get the values of the discriminant over the MEL_n.
Defining $\bar{\beta}$ as the value of β where $|\Delta(\phi_n(\beta), \beta)|$ has a critical point, and defining $\bar{\alpha} = \phi_n(\bar{\beta})$ we get the point where the maximum energy of the unstable pocket lays and the value of the discriminant in this point is

$$\Delta(\bar{\alpha}, \bar{\beta}) = \psi.$$

Adding a damping term $\delta\dot{y}$ to (2) and from Theorem 3, one can obtain the minimum δ needed to vanish the whole pocket

$$\delta = \frac{2}{T}\ln(\mu)$$

where $\mu = \max\{|\mu_1|, |\mu_2|\}$ and the characteristic multipliers of (2) in $(\bar{\alpha}, \bar{\beta})$ are

$$\begin{aligned} \mu_1 &= \frac{\psi + \sqrt{\psi^2 - 4}}{2} \\ \mu_2 &= \frac{\psi - \sqrt{\psi^2 - 4}}{2} \end{aligned}$$

■

Remark 16 *Notice that by definition $(\bar{\alpha}, \bar{\beta})$ lays on a corresponding MEL_n.*

Example

As is well known, solutions of periodic differential equation

$$\ddot{x} + (\alpha + \beta q(t))x = 0$$

can not be obtain in an analytical manner but for a few examples such as Meissner equation where $q(t)$ is a piecewise constant periodic function, Lamé equation where $q(t)$ is an Jacobi elliptic function or when $q(t)$ is a piecewise linear

[3]We call unstable pockets the zones between two coexistence points belonging to the same unstable areas. An Arnold tongue with m coexistence points have the same number of unstable pockets.

function, see [3] or [5].In this section we obtain an analytical approximation of the $MEL_n, n = 1, 2, \ldots,$, of the Meissner equation, and then, we use the lemma 14 in order to vanish an unstable pocket.
Consider the Meissner equation

$$\ddot{y} + (\alpha + \beta sign(\sin(t)))\, y = 0 \tag{14}$$

with $\alpha, \beta \in \mathbb{R}_+ \cup \{0\}$. Defining $x_1 = y$, $x_2 = \dot{y}$ and $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}'$, (14) could be rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -(\alpha + \beta sign(\sin(t))) & 0 \end{bmatrix} x \tag{15}$$

for sake of simplicity, we are only interested in the solution of (15) for $\alpha > \beta$, the monodromy matrix M for this parameter values is [14]

$$M = \begin{bmatrix} \cos(\pi w_2) & \frac{1}{w_2}\sin(\pi w_2) \\ -w_2\sin(\pi w_2) & \cos(\pi w_2) \end{bmatrix} \cdot \begin{bmatrix} \cos(\pi w_1) & \frac{1}{w_1}\sin(\pi w_1) \\ -w_1\sin(\pi w_1) & \cos(\pi w_1) \end{bmatrix}$$

where $w_1 = \sqrt{\alpha+\beta}$ and $w_2 = \sqrt{\alpha-\beta}$. We know that the characteristic multipliers of M depends on its trace $\Delta(\alpha, \beta)$, i. e.

$$\mu_{1,2} = \frac{\Delta(\alpha,\beta) \pm \sqrt{\Delta(\alpha,\beta)^2 - 4}}{2}$$

and $\Delta(\alpha, \beta)$ is monotonic in the intervals $\alpha \in [\xi_i, \xi_{i+1}]$ where ξ_i represent the zeros of $\frac{\partial}{\partial\alpha}\Delta(\alpha,\beta)$, so

$$\begin{aligned} \Delta(\alpha,\beta) &= 2\cos(\pi w_1)\cos(\pi w_2) \\ &\quad -\left(\frac{w_1}{w_2} + \frac{w_2}{w_1}\right)\sin(\pi w_1)\sin(\pi w_2) \\ &= \frac{(w_1+w_2)^2}{2w_1w_2}\cos(\pi w_1 + \pi w_2) \\ &\quad -\frac{(w_1-w_2)^2}{2w_1w_2}\cos(\pi w_1 - \pi w_2) \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial}{\partial\alpha}\Delta(\alpha,\beta) &= -\frac{\left(w_1^2 - w_2^2\right)^2}{4w_1^3w_2^3}\cos(\pi w_1 + \pi w_2) \\ &\quad +\frac{\left(w_1^2 - w_2^2\right)^2}{4w_1^3w_2^3}\cos(\pi w_1 - \pi w_2) \\ &\quad -\frac{\pi(w_1+w_2)^3}{4w_1^2w_2^2}\sin(\pi w_1 + \pi w_2) \\ &\quad -\frac{\pi(w_1-w_2)^3}{4w_1^2w_2^2}\sin(\pi w_1 - \pi w_2) \end{aligned} \tag{17}$$

if we choose values of α and β such that

$$\frac{w_1}{w_2} + \frac{w_2}{w_1} \approx 2$$

then (16) and (17) reduce to

$$\Delta(\alpha,\beta) \approx 2\cos(\pi w_1 + \pi w_2) \tag{18}$$

$$\frac{\partial}{\partial\alpha}\Delta(\alpha,\beta) \approx -\frac{\pi(w_1+w_2)}{w_1w_2}\sin(\pi w_1 + \pi w_2) \tag{19}$$

so, for (19) to be equal to zero, the condition

$$w_1 + w_2 = n$$

must be fulfilled; and the approximation of the maximum energy point for each n is

$$\phi_n(\beta) \approx \frac{\beta^2}{n^2} + \frac{n^2}{4} \tag{20}$$

for $n \in \mathbb{Z}_+$
Fig. 2 shows the transition curves of the Meissner equation, the approximation of MELs obtained by (20) and MELs obtained numerically. From Fig. 2 one can notice that the approximation (20) is good enough to use in order to show some of its properties. One can see that each MEL_n goes through the coexistence points in the unstable zones.

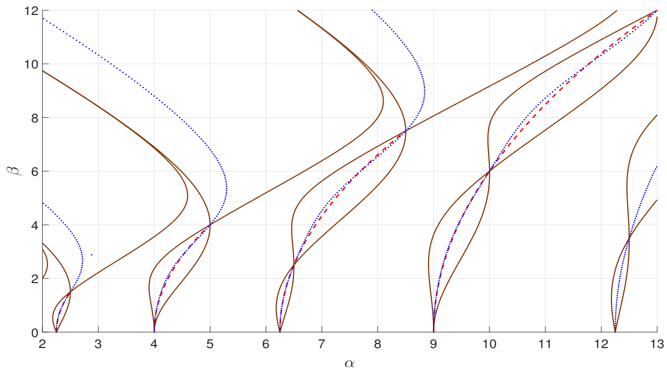


Figure 2: Transition curves in black, Approximation of $MELs$ in red and numerical approximation of $MELs$ in blue dotted line. For the Meissner equation $\ddot{x} + (\alpha + \beta sign(cos(t)))x = 0$

Substituting (20) into (16) one gets the value of $\Delta(\alpha, \beta)$ over the MEL_n:

$$\Delta\left(\phi_n\left(\beta\right), \beta\right) = \frac{8\beta^2 \cos\left(\frac{2\pi}{n}\beta\right) - 2n^4\left(-1\right)^n}{4\beta^2 - n^4}$$

Fig. 3 shows $\Delta\left(\phi_5\left(\beta\right), \beta\right)$, that is, the discriminant value over the MEL_5. From Fig. 3 one can notice that $\left|\Delta\left(\phi_n\left(\beta\right), \beta\right)\right| = 2$ at three different values of β, it means that MEL_5 goes though two coexistence points of the fifth unstable region which rises from $\alpha = 6.25$, $\beta = 0$. The function $\Delta\left(\phi_5\left(\beta\right), \beta\right)$ has two critical values more, these represent the energy's maximum value of each unstable pocket. If we are able to calculate the exact β on these critical points, we will be capable of vanish the unstable pocket associated to each critical value.

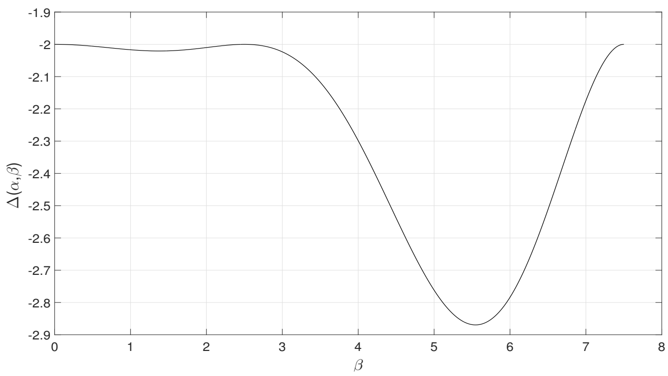


Figure 3: Discriminant of Meissner equation, over the approximation of MEL_5 i.e. $\Delta(\phi_5(\beta), \beta)$.

The critical values of β associated to coexistence points are $\beta_1 = 0$, $\beta_2 = 2.5$ and $\beta_3 = 7.5$. And those associated to the energy's maximum value of each unstable pocket are $\bar{\beta}_1 = 1.3754$ and $\bar{\beta}_2 = 5.5462$, so the points, inside the unstable zone rising in $\alpha = 6.25$, $\beta = 0$, where the maximum energy is located are approximately: $(6.325, 1.3754)$ and $(7.480, 5.5462)$. The maxim eigenvalue associated to each maximum energy point are:

$$\begin{aligned} |\mu_{1,\max}| &= 1.1546 \\ |\mu_{2,\max}| &= 2.4636 \end{aligned}$$

By lemma 14, if one adds a dissipative term $\delta\dot{y}$ with the enough amount of damping δ to (14) then, some unstable zones would disappear. The approximation of the minimum amount of dissipation needed to vanish the first and second unstable pocket of the unstable region rising in $\alpha = 6.25$, $\beta = 0$ are

$$\begin{aligned} \delta_1 &= 0.0456 \\ \delta_2 &= 0.2870 \end{aligned}$$

Using the numerical calculation of the MEL_5 we have obtain that minimum amount of dissipation needed to vanish the pockets are

$$\begin{aligned} \delta_1 &= 0.0465 \\ \delta_2 &= 0.2921 \end{aligned}$$

Fig. 4 shows Arnold tongues for damped Meissner equation with $\delta = 0.0456$, one can see that there is a small unstable pocket near the point $(6.325, 1.38)$, this is due to the approximation made in (20). If we use the damping coefficient $\delta = 0.0465$ instead, then, the whole pocket will vanish.

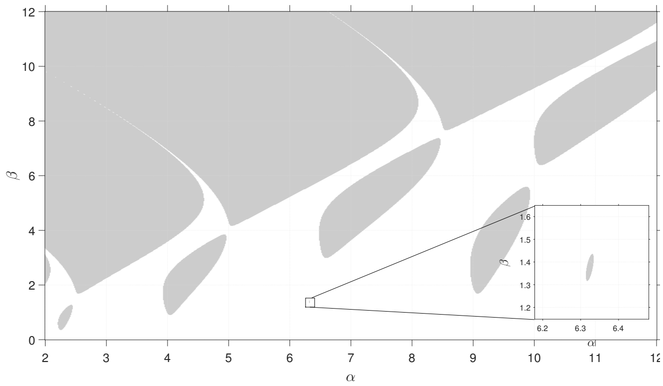


Figure 4: Stable and unstable regions for damped Meissner equation with $\delta = 0.0456$

This same method can be used in order to vanish any unstable pocket of any Hill equation.

Remark 17 *It is clear that if we eliminate some pocket in the nth Arnold tongue, all the pockets on the same tongue "below" it, also will disappear.*

Conclusion

By using the new definitions, Maximum Energy Points and Maximum Energy Lines, we propose a method to vanish any unstable pocket of any Arnold tongue by simply adding a dissipative term $\delta\dot{x}$ to the Hill equation; we have given a formula to obtain the minimum damping δ needed to vanish the whole pocket. Also, we give a new characterization of the coexistence points in terms of the maximum energy line.
The main problem with the method, here introduced, is that in order to obtain the maximum energy of a pocket, i.e. the value of α and β where a solution has the maximum growth within all the solutions of an unstable pocket, we need to obtain the corresponding MEL_n which is, in almost all the cases, only achieved by the numerical calculation of the characteristic multipliers of the system.

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