Stability of Capillary Waves of Finite Amplitude

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Summary. The direct Lyapunov method is used to prove the stability of the exact Crapper solution for capillary waves. The dynamic equations of the capillary wave are presented in the form of an infinite Euler-Lagrange chain of equations for the Stokes coefficients. The stationary solution found for these equations is the Crapper solution for capillary waves. With the help of energy and momentum conservation laws the Lyapunov function is constructed. It is shown that the Lyapunov function is positive definite with respect to any perturbations of waves surfaces, for waves with the period multiple of wavelength.

Introduction

In [1] the exact solution of the problem of the potential plane-parallel flow of an ideal fluid in the domain $-\infty < x < \infty$, $-\infty < y < \eta(kx)$ was constructed, where the function $\eta(kx)$ is periodic $\eta(kx) = \eta(kx + 2\pi)$, the wave number k is related to the wave length as follows $\lambda = 2\pi/k$. The Laplace condition $p - p_0 + \sigma/r = 0$ is satisfied on the wave surface $\eta(kx)$, where r is the curvature radius of the cylinder, p and p_0 are the fluid pressure inside and outside the cylinder, σ is the surface tension coefficient.

In this work we present the analytic solution for the capillary waves stability problem (earlier numerical methods were used). We use the second variation of the Lyapunov function to prove the stability of capillary waves with respect to symmetric and non-symmetric disturbances.

Stationary capillary waves

To describe the dynamics of capillary waves, we use the wave parametrization, introduced by Stokes [2]. We seek the conformal mapping of the disc $|\zeta| < 1$ of the complex plane ζ with a cut on the positive part of abscissa on the domain of one wave period on the complex plane $z = x + iy$ in the following form

$$z(\zeta) = \frac{\lambda}{2\pi}\left[i\ln\zeta + \sum_{n=1}^{\infty} z_n\zeta^n\right] \quad . \tag{1}$$

The circle $\zeta = e^{i\gamma}$ corresponds to the surface of the wave $z = x_s + i\eta$. We consider the real and imaginary parts of the Laurent series coefficients $z_n = x_n + iy_n,\ n = 1, 2, \dots$ to be the generalized coordinates of the wave $q_i,\ i = 1, 2, \dots$.

The kinetic energy of the wave is the quadratic function of generalized velocities $\dot{x}_0,\ \dot{q}_i,\ i = 1, 2, \dots$, where x_0 is the cyclic coordinate that determines the horizontal movement of the wave, $\dot{x}_0$ – the wave propagation velocity.

The summands in the kinetic energy may be separated into three groups: quadratic in $\dot{x}_0$, linear in x_0 and independent of $\dot{x}_0$

$$E_{\text{kin}} = \frac{1}{2}M\dot{x}_0^2 + M_1\dot{x}_0 + M_2 = \frac{(M\dot{x}_0 + M_1)^2}{2M} + M_*, \quad M_* = M_2 - \frac{M_1^2}{2M} \quad . \tag{2}$$

Here M is independent of velocities, M_1 and M_2 are the linear and quadratic function of velocities $\dot{q}_i$. As E_{kin} is positively definite, then M_* is also a positively definite quadratic form of $\dot{q}_i$.

Suppose that the system of Lagrange equations has a stationary solution, for which $\dot{x}_0 = u, \quad \dot{q}_i = 0, \quad i = 1, 2, \dots$. In this solution the surface of the wave moves with velocity u, without changing its form.

For stationary motion $M_* = 0$ and, thus, the energy value is $\frac{(M_0 u)}{2} + E^0_{\text{pot}} = E_0$, where E^0_{pot} is the value of potential energy at a stationary point. The function E is a Lyapunov function if it is positively definite. As M_* is positively definite, we consider only the functional $U = \frac{(M_0 u)^2}{2M} + E_{\text{pot}}$, If the stationary point is the minimum of U, the Lyapunov Theorem implies that the stationary motion is stable.

The Lyapunov function can be expressed in the dimensionless form as follows

$$U = \sigma\frac{\lambda}{2\pi}\bar{U}, \quad \bar{U} = \frac{S_0^2}{4S}c^2 + \bar{l}, \quad \bar{l} = \frac{l}{\lambda} = 1 + \sum_{k=1}^{\infty}|q_n|^2 \ , \tag{3}$$

where $\bar{U}$ and $\bar{l}$ is a dimensionless Lyapunov function and the arc length of one wave period, $S = \sum_{n=1}^{\infty} n(x_n^2 + y_n^2)$, S_0 is the value of S at the stationary point and c is the dimensionless wave velocity.

The assertion that the first variation of $\bar{U}$ equals zero allows us to find the parameters q_n of the wave and its propagation velocity c.

The solution of the variational equation $\delta\bar{U} = 0$ may be presented as follows

$$q_i = 2b^i \ , \tag{4}$$

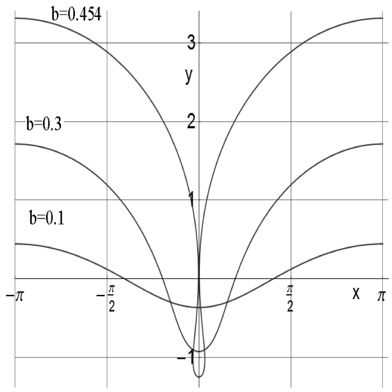

Figure 1: Capillary waves at different values of parameter b.

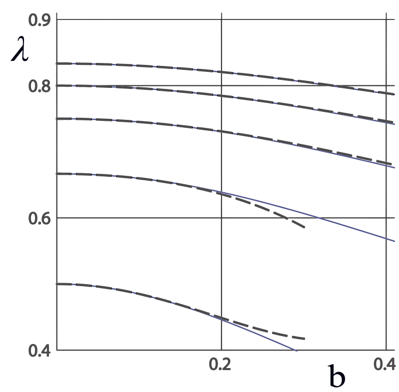

Figure 2: Eigenvalues (symmetric disturbances).

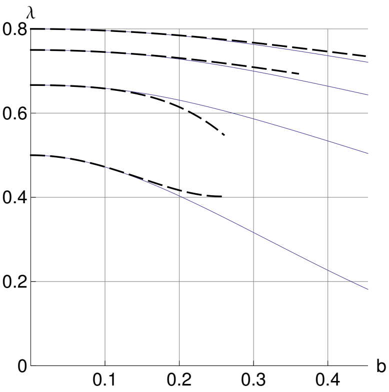

Figure 3: Eigenvalues (non-symmetric disturbances).

where b is a parameter of a family of solutions. To prove this we consider small disturbances of coordinates with respect to stationary values

$$q_n = 2b^n + \varepsilon(\xi_n + i\eta_n), \quad n = 1, 2, \dots \ . \tag{5}$$

We substitute them into function $\bar{l}$ (3) and expand by parameter ε. Then we will find the expansion for the Stokes coefficients x_n and y_n, for functional S and for function U:

$$\bar{l} = 1 + \sum_{k=1}^{\infty} \left((2b^n + \varepsilon\xi_n)^2 + \varepsilon^2 \eta_n^2 \right) = 1 + \varepsilon\delta\bar{l} + \varepsilon^2\delta^2\bar{l} \ , \tag{6}$$

$$S = S_0 + \varepsilon\delta S + \varepsilon^2\delta^2 S \ , \quad \bar{U} = U_0 + \varepsilon\delta U + \varepsilon^2\delta^2 U \ ,$$

where δ and δ^2 denote the first and second variation accordingly. Then we calculate the first variation of U, considering (4). Then we obtain the same expressions for x and y as did Crapper. Therefore, a new deduction method for the known exact solution for the capillary wave [1] is presented. The values $b = b_0 = 0.454,\ a = 2.280$ corresponds to the maximum wave development. In Fig. 1 the graphs of waves with values $b := 0.1;\ 0.3$ and maximum wave development $b = 0.454$ are presented.

Second Variation

The second variation is $\delta^2\bar{U} = \frac{1}{2}\frac{d^2\bar{U}}{d\varepsilon^2}\Big|_{\varepsilon=0}$ is the quadratic form of variations $\xi_i,\ \eta_i,\ i = 1, 2, \dots$. It is expressed through the first and second variations of functionals S and $\bar{l}$. The variables ξ_n and η_n of the second variation $\delta^2\bar{U}$ may be separated and the second variation $\delta^2\bar{U}$ may be presented as the sum of two quadratic forms $\delta^2\bar{U} = \delta^2\bar{U}_1(\xi) + \delta^2\bar{U}_2(\eta)$. The first $\delta^2\bar{U}_1(\xi)$ depends only on ξ and is expressed through $\delta^2 S_1$ and $\delta^2\bar{l}_1$, which depend only on ξ. The second is expressed through $\delta^2 S_2$ and $\delta^2\bar{l}_2$, which depend only on η.

The first quadratic form defines the stability of the wave with respect to the symmetric disturbances ξ_n, the second one defines the stability with respect to the asymmetric disturbances η_n.

Let us first consider the quadratic forms of second variations for symmetric disturbances

$$\delta^2\bar{U}_1 = \frac{c^2}{4}\left(\frac{(\delta S)^2}{S_0} - \delta^2 S_1\right) + \delta^2\bar{l}_1 \ .$$

For $\delta^2 U_1(\xi)$ the following inequality holds $\delta^2\bar{U}_1 > \lambda_{\min}\sum_{n=1}^{\infty}(\xi_n)^2$, where $\lambda_{\min}$ is the smallest eigenvalue of the quadratic form. Thus, the inequality above implies that the second variation $\delta^2 U > 0$ is strictly positive for all variations δq_i. By the Lyapunov Theorem the stationary motion of capillary wave is stable for all possible amplitude values.

The eigenvalues determine the main oscillation frequencies near stationary motion.

Consider now the quadratic form of the second variation for non-symmetric disturbances

$$dU2 \ \ \delta^2\bar{U}_2 = -\frac{c^2}{4}\delta^2 S_2 + \delta^2\bar{l}_2 \ . \tag{7}$$

The matrix $b_{mn} = \frac{1}{2}\frac{\partial^2(\delta^2\bar{U}_2)}{\partial\eta_m\partial\eta_n}$ for $b = 0$ is diagonal and $b_{nn} = (n-1)/n,\ n = 2, 3, \dots$. The eigenvalues are $\lambda_n = b_{nn}$. The matrix b_{mn} is singular, its determinant equals zero.

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References

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