Vibration decay and positioning time of sampled-data systems with dry friction

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Summary. In this paper, we present the stabilization effect of dry friction on an otherwise unstable system where the instability is caused by the sampled-data nature of the applied position controller. It is illustrated by analytical and simulation results, that vibrations close to the stability limit have a special concave envelope, and a closed form expression is provided to predict the positioning time as function of the mechanical and control parameters.

Introduction

In robotic applications the dominant sources of dissipation are the actuators, bearings and the motion transmission elements in the drive-train. In the analysis of these systems, the effect of friction is often neglected due to the application of a suitable friction compensation algorithm [1], or, because this assumption results in a more conservative stability condition [3]. However, dry friction can have an important effect on positioning time and accuracy, and it can cause interesting vibrations. For example, it was observed in [2] that vibrations in sampled-data systems can have a special concave envelope which is qualitatively different from that of continuous-time systems with viscous or dry friction [2].

Effective model of position control

In order to illustrate the effect of dry friction on positioning, the example of a rotating disk driven by a DC motor is considered, where the zero reference position is regulated by a discrete-time proportional controller with zero-order-hold signal reconstruction for the commanded motor torque. The corresponding equation of motion can be written as

$$J\ddot{\varphi}(t) + \tau_{\mathrm{C}}\,\mathrm{sgn}\left(\dot{\varphi}(t)\right) = -k_{\mathrm{p}}\varphi(t_j), \quad t \in [t_j, t_j + \theta), \quad t_j = j\theta, \quad j = 0, 1, 2\ldots, \tag{1}$$

where $\varphi(t)$ represents the angular position of the disk as a function of time, J denotes the combined second moment of inertia of the disk and the rotor, τ_{C} is the magnitude of the assumed Coulomb friction torque and k_{p} is the proportional control gain. In addition, t_j denotes the jth sampling instant, θ is the sampling time and $\varphi(t_j)$ denotes the sampled position at the beginning of the jth time interval according to the zero-order-hold.
Equation (1) forms a non-homogeneous differential equation between two sampling instants assuming that the direction of motion does not change. It can be solved for the discrete state variables collected in $\mathbf{z}_j = [\varphi(t_j)\ \dot{\varphi}(t_j)]^{\mathrm{T}}$ in the form of a non-homogeneous discrete map $\mathbf{z}_{j+1} = \mathbf{A}\mathbf{z}_j - \mathbf{a}\,\mathrm{sgn}\left(\dot{\varphi}(t)\right)$. Based on the characteristic equation of the lead matrix $\mathbf{A}$, the characteristic multipliers $\mu_{1,2}$ can be determined which represents the dynamic behaviour of the undamped system. If $0 < p < 16$, then $\mu_{1,2} = \rho\exp(\pm i\vartheta)$. When the parameter p is in the range presented above, then $1 < \rho < 3$ which means the frictionless system will have exponentially unstable oscillations around the reference position. Therefore, in this parameter range, the motion can be characterized as a damped oscillator where the "viscous" damping term is selected to be *negative* to model the unstable behavior. Neglecting the higher harmonics due to sampling, an effective continuous-time oscillator model can be derived in the form

$$\ddot{\varphi}(t) + f_0\omega_{\mathrm{n}}^2\,\mathrm{sgn}\left(\dot{\varphi}(t)\right) = -\omega_{\mathrm{n}}^2\varphi(t) + 2\zeta\omega_{\mathrm{n}}\dot{\varphi}(t), \quad \text{with} \quad \omega_{\mathrm{n}} = \frac{\sqrt{\ln^2(\rho)+\vartheta^2}}{\theta}, \quad \zeta = \frac{\ln(\rho)}{\sqrt{\ln^2(\rho)+\vartheta^2}} \tag{2}$$

where parameter $f_0 = \tau_{\mathrm{C}}/k_{\mathrm{p}}$ describes the same sticking region as that of the original sampled-data system.

Analysis of the motion

For the solution of Eq. (2) when, the dominant damping ratio is considered to be in the range $0 < \zeta < 1$, and the initial conditions are selected as $\varphi(0) = \varphi_0 > 0$ and $\dot{\varphi}(0) = 0$. With these, the solution can be determined until the first velocity reversal which happens at $t = T/2 = \pi/\omega_{\mathrm{d}}$ with $\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\zeta^2}$ in the form

$$\varphi^-(t) = A_0\mathrm{e}^{\zeta\omega_{\mathrm{n}}t}\left(\cos(\omega_{\mathrm{d}}t) - \kappa\sin(\omega_{\mathrm{d}}t)\right) + f_0, \quad \text{with} \quad A_0 = \varphi_0 - f_0 \quad \text{and} \quad \kappa = \zeta\omega_{\mathrm{n}}/\omega_{\mathrm{d}}. \tag{3}$$

By looking for a periodic solution, the Coulomb friction stabilizes the unstable motion due to the sampling such that an unstable limit cycle exist. Based on Eq. (3) when $\varphi^-(T/2) = -\varphi_0$, the corresponding critical initial position can be determined as

$$\varphi_0 = f_0\frac{\mathrm{e}^{\kappa\pi}+1}{\mathrm{e}^{\kappa\pi}-1} = f_0\coth\left(\frac{\kappa\pi}{2}\right). \tag{4}$$

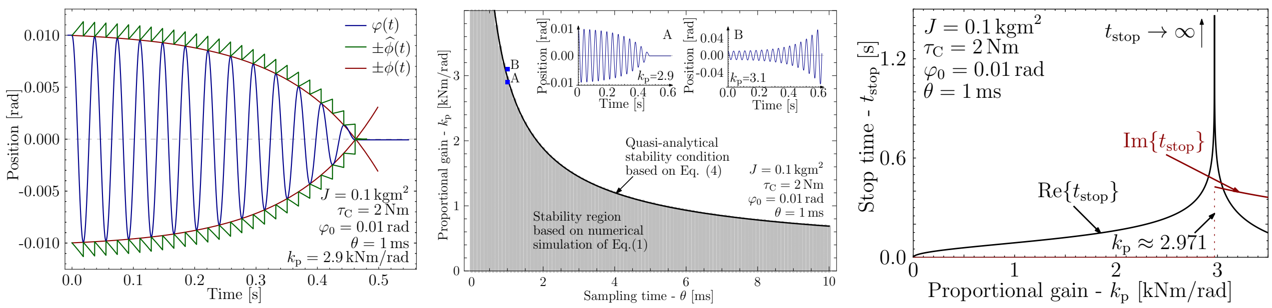

Figure 1: Special concave shape vibration

When the direction of motion changes and the velocity becomes positive, after some trigonometric manipulation, the solution for the next half-period of the oscillation becomes

$$\varphi^{+}(t) = \frac{A_1}{\sqrt{1-\zeta^2}}\,\mathrm{e}^{\zeta\omega_\mathrm{n}t}\cos(\omega_\mathrm{d}t + \arcsin\zeta) - f_0, \quad \text{with} \quad A_1 = A_0 - 2f_0\mathrm{e}^{-\kappa\pi}. \tag{5}$$

The consecutive piecewise segments of the solution can be combined in a special closed form

$$\varphi(t) = \frac{A_n}{\sqrt{1-\zeta^2}}\,\mathrm{e}^{\zeta\omega_\mathrm{n}t}\cos(\omega_\mathrm{d}t + \arcsin\zeta) + (-1)^n f_0 \quad \text{with} \quad n = \lfloor 2t/T \rfloor \tag{6}$$

where

$$A_n = A_0 - 2f_0\sum_{k=1}^{n}\varepsilon^k,\ n = 1, 2, \dots, \quad \text{with} \quad \varepsilon = \mathrm{e}^{-\kappa\pi}. \tag{7}$$

This solution with specific system parameters is presented as the solid blue curve in Fig. 1. In Eq. (6), the term multiplying the cosine function gives the amplitude decay, and considering a positive offset with f_0, the upper concave vibration envelope is approximated by the function

$$\widehat{\phi}(t) = \frac{A_n}{\sqrt{1-\zeta^2}}\,\mathrm{e}^{\zeta\omega_\mathrm{n}t} + f_0, \quad \text{with} \quad A_n = A_0 - 2f_0\sum_{k=1}^{n}\varepsilon^k. \tag{8}$$

It is presented as the green sawtooth shape function in Fig. 1 which shows that every second local minimum of this function sits exactly on the vibration peaks. These local minima can be connected by a continuous function by removing the floor function and calculating the sum in Eq. (8) as $\varepsilon(\varepsilon^{\omega_\mathrm{d}t/\pi} - 1)/(\varepsilon - 1)$. Then the approximated vibration envelope is described by $\pm\phi(t)$ with

$$\phi(t) = \frac{A(t)}{\sqrt{1-\zeta^2}}\,\mathrm{e}^{\zeta\omega_\mathrm{n}t} + f_0 \quad \text{where} \quad A(t) = A_0 - 2f_0\frac{\mathrm{e}^{-\zeta\omega_\mathrm{n}t}-1}{1-\mathrm{e}^{\kappa\pi}}. \tag{9}$$

The corresponding lower and upper envelope segments are shown in red in Fig. 1. This figure also shows that the motion will stop in finite time. By using the stop conditions $\phi = 0$, the positioning time can be estimated from above as

$$t_\mathrm{stop} = \frac{1}{\zeta\omega_\mathrm{n}}\ln\left(\frac{2f_0 - f_0\sqrt{1-\zeta^2}(1-\mathrm{e}^{\kappa\pi})}{2f_0 + A_0(1-\mathrm{e}^{\kappa\pi})}\right). \tag{10}$$

The middle and right charts in Fig. 1 show the maximum attainable control gain and the corresponding positioning time. It can be seen, that the positioning time is rapidly increasing near to the stability boundary. The trade-off between fast positioning and accuracy can be analyzed by using these charts and/or Eqs. (4) and (10).

Conclusions

In this paper, the vibration decay characteristics of position controlled mechanical systems were investigated by considering dry friction as the primary source of physical dissipation. For the analysis of the digitally controlled system, an effective continuous-time model was derived and it was used to explain the concave envelope vibrations. The vibration envelope was closely approximated by a closed form amplitude decay function, which made it also possible to determine the positioning time as function of the control and system parameters. As no special assumptions were used in constructing the piecewise solution of the equation of motion regarding the type of friction, the results naturally generalize to the case when both viscous damping and dry friction are present.

References

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