On the Kukles cubic system

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Summary. Using our bifurcational geometric approach, we solve the problem on the maximum number and distribution of limit cycles in the Kukles system representing a planar polynomial dynamical system with arbitrary linear and cubic right-hand sides and having an anti-saddle at the origin. We also apply alternatively the Wintner–Perko termination principle to solve this problem.

Global bifurcations of limit cycles

We study the Kukles cubic system

$$\dot{x} = y, \quad \dot{y} = -x + \delta y + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3. \tag{1}$$

I. S. Kukles was the first who began to study (1) solving the center-focus problem for this system in 1944: he gave the necessary and sufficient conditions for $O(0,0)$ to be a center for (1) with $a_7 = 0$ [8]. Later, system (1) was studied by many mathematicians. For example, in [9] the necessary and sufficient center conditions for arbitrary system (1), when $a_7 \neq 0$, were conjectured. In [11], global qualitative pictures and bifurcation diagrams of a reduced Kukles system $(a_7 = 0)$ with a center were given. In [12], the global analysis of system (1) with two weak foci was carried out. In [13], the number of singular points under the conditions of a center or a weak focus for (1) was investigated. In [14], new distributions of limit cycle for the Kukles system were obtained. In [10], an accurate bound of the maximum number of limit cycles in a class of Kukles type systems was provided.

In [2], we constructed a canonical cubic dynamical system of Kukles type and carried out the global qualitative analysis of a special case of the Kukles system corresponding to a generalized cubic Liénard equation. In particular, it was shown that the foci of such a Liénard system could be at most of second order and that such system could have at most three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system were studied. As a result, a classification of all possible types of separatrix cycles for the generalized cubic Liénard system was obtained and all possible distributions of its limit cycles were found.

Applying Erugin's two-isocline method [1] and studying the rotation properties of all the parameters of (1), we prove the following theorem.

Theorem 1. *Kukles system* (1) *with limit cycles can be reduced to the canonical form*

$$\dot{x} = y, \quad \dot{y} = q(x) + (\alpha_0 - \beta + \gamma + \beta\, x + \alpha_2\, x^2)\, y + (c + dx)\, y^2 + \gamma\, y^3, \tag{2}$$

where
1) $q(x) = -x + (1 + 1/a)\, x^2 - (1/a)\, x^3,\ a = \pm 1, \pm 2$ *or*
2) $q(x) = -x + b\, x^3,\ b = 0, -1,$ *or*
3) $q(x) = -x + x^2;$
$\alpha_0, \alpha_2, \gamma$ *are field rotation parameters and* β *is a semi-rotation parameter.*

Using system (2) and studying global bifurcations of its limit cycles, by means of our bifurcational geometric approach developed in [3]–[7], we prove the following theorem.

Theorem 2. *Kukles cubic system* (1) *can have at most four limit cycles in* $(3\!:\!1)$*-distribution.*

Application of the Wintner–Perko termination principle

For the global analysis of limit cycle bifurcations in [1], we used the Wintner–Perko termination principle which connects the main bifurcations of limit cycles. Let us formulate this principle for the polynomial system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{3}$$

where $\boldsymbol{x} \in \mathbf{R}^2;\ \boldsymbol{\mu} \in \mathbf{R}^n;\ \boldsymbol{f} \in \mathbf{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function).

Theorem 3 (Wintner–Perko termination principle). *Any one-parameter family of multiplicity-*m *limit cycles of relatively prime polynomial system* (3) *can be extended in a unique way to a maximal one-parameter family of multiplicity-*m *limit cycles of* (3) *which is either open or cyclic.*

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of (3), *which is typically a fine focus of multiplicity* m, *or on a* $($*compound*$)$ *separatrix cycle of* (3), *which is also typically of multiplicity* m.

The proof of the Wintner–Perko termination principle for general polynomial system (3) with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \tag{4}$$

with a single parameter $\lambda \in \mathbf{R}$; see [1]. In particular, if λ is a field rotation parameter of (3), it is valid the following Perko's theorem on monotonic families of limit cycles.

Theorem 4. *If L_0 is a nonsingular multiple limit cycle of (4) for $\lambda = \lambda_0$, then L_0 belongs to a one-parameter family of limit cycles of (4); furthermore*:
1) *if the multiplicity of L_0 is odd, then the family either expands or contracts monotonically as λ increases through λ_0*;
2) *if the multiplicity of L_0 is even, then L_0 bifurcates into a stable and an unstable limit cycle as λ varies from λ_0 in one sense and L_0 disappears as λ varies from λ_0 in the opposite sense; i. e., there is a fold bifurcation at λ_0.*

Using Theorems 3 and 4, we give an alternative proof of Theorem 2 for system (1), namely, we prove the following theorem.

Theorem 5. *There exists no system (1) having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, system (1) cannot have either a multiplicity-four limit cycle or four limit cycles around a singular point, and the maximum multiplicity or the maximum number of limit cycles surrounding a singular point is equal to three. Moreover, system (1) can have at most four limit cycles with their only possible $(3\!:\!1)$-distribution.*

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