Stationary and Non-stationary Dynamics of the Parametric Pendulum

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Summary. The nonlinear dynamics of a parametrically excited pendulum is addressed. The proposed analytical approach is conceived in order to describe the pendulum dynamics beyond the simplified regimes usually considered in literature, where stationarity and small amplitude oscillations are assumed. By combining complexification and Limiting Phase Trajectory (LPT) concepts the pendulum dynamics in the neighborhood of the main parametric resonance is investigated. At first, the non-stationary dynamics in the quasi-linear approximation is considered; afterwards, both stationary and non-stationary regimes are studied without any restriction on the pendulum oscillation amplitudes. The identification of the bifurcations of the stationary states as well as the large-amplitude corrections of the stability thresholds emanating from the main parametric resonance are the main results provided by the proposed study.

Introduction and governing equations

Parametrically excited systems represent a class of widely studied problems in nonlinear dynamics [1]. In this context the parametrically excited pendulum is here considered, the motion of which is described by the following equation

$$\frac{d^2q}{dt^2} + \left[1 + A_1 \sin\left(\frac{\omega}{\omega_0}t\right)\right] \sin q = 0, \tag{1}$$

in which time $\tilde{t} = \omega_0 t$, ω_0 being the pendulum linear natural frequency and the tilde being omitted. We set $\omega_1 = \frac{\omega}{\omega_0}$, where ω is the frequency of the parametric excitation.

Non-stationary dynamics in the quasi-linear approximation

At first the non-stationary dynamics of the quasi-linear approximation of (1) is addressed. Towards this goal, the series expansion of the sine function is introduced. By considering that, $q = \mathcal{O}(\varepsilon^{1/2})$ and $A_1 = \varepsilon A$, with $\varepsilon << 1$, equation (1) can be rewritten as

$$\frac{d^2q_0}{dt^2} + (1 + 4\varepsilon A \sin\omega_1 t)q_0 + 8\varepsilon\alpha q_0^3 = 0 \tag{2}$$

The complex functions $\psi = \frac{1}{\sqrt{2}}(\dot{q_0} + iq_0)$ and $\psi^* = \frac{1}{\sqrt{2}}(\dot{q_0} - iq_0)$ are now introduced [2]. The main parametric resonance (1:2) condition is assumed and the change of variable $\psi = \varphi(t)e^{it}$ is considered. After some algebraic manipulations, the slow time $t_1 = \varepsilon t$ is introduced and the solution, expressed by $\varphi(t, t_1) = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 \ldots$, is truncated to the first order approximation leading to the principal approximation equation in slow time:

$$\frac{d\varphi_0}{dt_1} - 3i\alpha|\varphi_0|^2\varphi_0 - e^{-i\beta t_1}A\varphi_0^* = 0, \tag{3}$$

in which $\varphi_0 = \varphi_0(t_1)$ and $\omega - 2\omega_0 = \beta\varepsilon\omega_0$; thus, β represents the detuning with respect to the main parametric resonance. By considering the further change of variables $\varphi_0 = \Phi_0 e^{-i\beta t_1}$ and $\Phi_0 = a(t_1)e^{-i\delta(t_1)}$, with $a(t_1), \delta(t_1) \in \mathbb{R}$, an Hamiltonian system for the variables $a(t_1), \delta(t_1)$ is obtained, the phase trajectories of which can be represented in the (a-δ) plane by the equation

$$\beta a^2 - 3\alpha a^4 - 2a^2 \sin 2\delta = \text{const.} \tag{4}$$

in which $A = 1$ is assumed. In Figure 1 the phase plane stemming from (4) is shown; the stationary points are identified by the conditions $a^2 = (\beta \pm 2)/6\alpha$ for $\delta = 3/4\pi$ and $\delta = \pi/4$, respectively. The first type is stable and exists for $\beta \geq -2$, whereas the second one is unstable and it occurs only for $\beta \geq 2$. The Limiting Phase Trajectory (LPT), encircling all trajectories, corresponds to the most intensive energy taking off by the pendulum from the parametric excitation energy source (at given initial conditions) in the main parametric resonance condition. Therefore it can be noticed that, while near the main parametric resonance (Fig.1a), large LPTs occurs, small LPTs, implying strong localization, arise exiting from the resonance zone (Fig.1b).

Large-amplitude stationary and non-stationary dynamics

In (1) the complex amplitude of the pendulum oscillations is introduced according to

$$\psi = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\omega_1/2}}\frac{dq}{dt} + i\sqrt{\omega_1/2}q\right), \quad q = \frac{-i}{\sqrt{\omega_1}}(\psi - \psi^*), \quad \frac{dq}{dt} = \sqrt{\frac{\omega_1}{4}}(\psi + \psi^*), \tag{5}$$

We set now $\psi = \varphi e^{i\omega_1/2\,t}$. Then, multiplying by $e^{-i\omega_1/2\,t}$ and integrating with respect to the "fast" time t, the condition providing elimination of resonance (secular) terms with $\tau = \beta t$ is obtained as:

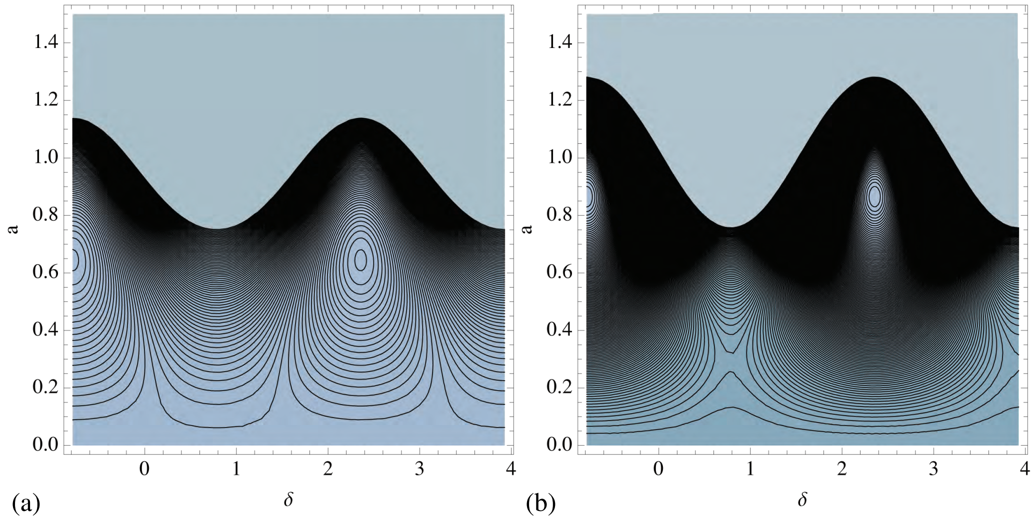

Figure 1: Phase portrait δ-a in the range $-\pi/4 < \delta < 5/4\pi$. (a) $\beta = 0.5$, $\alpha = 1.0$ with stable stationary points at $\delta = -\pi/4, 3/4\pi$; (b) $\beta = 2.5$, $\alpha = 1.0$ with the newborn unstable stationary point at $\delta = \pi/4$.

$$is\frac{\partial\varphi}{\partial\tau} - \frac{\omega_1}{4}\varphi + \frac{1}{\sqrt{\omega_1}}J_1\left(\sqrt{\frac{4}{\omega_1}}|\varphi|\right)\frac{\varphi}{|\varphi|} = -\frac{A}{2\omega_1}J_1\left(\sqrt{\frac{4}{\omega_1}}|\varphi|\right)\frac{\varphi}{|\varphi|}e^{i\tau}, \tag{6}$$

By recalling that $\varphi = \sqrt{X}e^{i\tau}$ and that $X = \frac{\omega_1}{4}Q^2$, we get

$$\beta\omega_1 - \frac{\omega_1^2}{4} + \frac{2}{Q}J_1(Q) + \frac{A}{Q\sqrt{\omega_1}}J_1(Q) = 0. \tag{7}$$

On the basis of the above developed stationary dynamics analysis, the non-stationary one [3, 4] can be addressed within the same analytical framework used in the quasi-linear approximation.

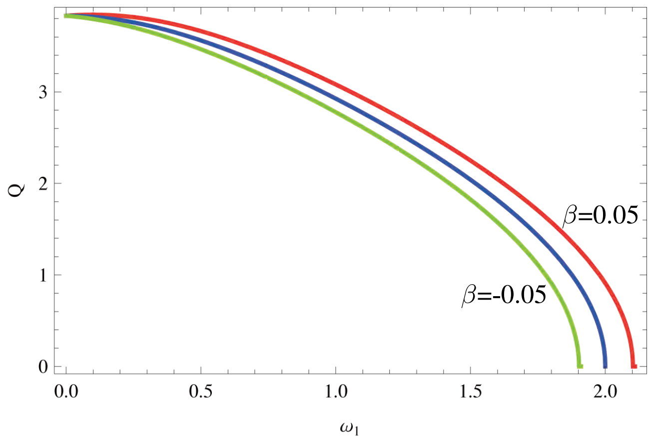

Figure 2: Relationship between ω_1 and Q for varying values of main parametric resonance detuning, $\beta = -0.05, 0.0, 0.05$, and $A = 0$.

Conclusions

In this study we present an analytical framework enabling to describe the nonlinear dynamics of a parametrically excited pendulum. In particular, by combining complexification and Limiting Phase Trajectory concepts the non-stationary dynamics for arbitrary pendulum oscillation amplitude can be described. In the quasi-linear approximation, the conditions for the appearance of unstable stationary regimes was determined; for large oscillations, the frequency-amplitude dependence for parametrically forced pendulum was described. On going work is devoted to complete the non-stationary analysis for the large amplitude oscillations case.

References

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