Harmonic balance method with iterative frequency technique for nonlinear oscillators

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Summary. The standard harmonic balance method (HBM) approaches periodic solutions of a nonlinear oscillator by calculating the amplitudes of harmonics with a given frequency. This method has difficulties in term of convergence when there are different periodic solutions for the same given frequency. This article presents an iteration procedure to calculate frequencies corresponding a given amplitude of the oscillator first harmonic. The higher order harmonic amplitudes are calculated by combining this procedure and HBM. The numerical applications show that this technique can be used to two periodic solutions corresponding to each amplitude of the first harmonic, including the "unstable" solution.

Formulations

The original principle of harmonic balance method is to express the periodic solution in terms of Fourier series with limited numbers of harmonics and to substitute this expression to the dynamic equation in order to find out balance of all harmonics. A typical difficulty of this method is linked to the dependence of the quality of the approach on the way to carry enough terms in the solution and check the order of the coefficients for all the neglected harmonics [1, 2]. A recent article [3] presents an iteration procedure for HBM which depends on exciting frequency in order to compute the periodic solution. This study developped this method with a new approach: calculating the exciting frequency to obtain an solution with a given amplitude. Let's consider a nonlinear oscillator given by

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \beta \frac{\mathrm{d}u}{\mathrm{d}t} + \omega_0^2 u + f(u) = F \cos \Omega t \tag{1}$$

The harmonic balance method propose for the periodic solution as follows

$$u = \frac{q_0}{2} + \sum_{n=1}^{N} q_n \cos n\Omega t + p_n \sin n\Omega t \tag{2}$$

Then, by substituting this approximation into equation (1) and by performing the Fourier series expansion, we obtain

$$\begin{cases} -n^2\Omega^2 q_n + \beta\Omega p_n + \omega_0^2 q_n + C_n = F\delta_{1n} \\ -n^2\Omega^2 p_n - \beta\Omega q_n + \omega_0^2 p_n + S_n = 0 \end{cases} \tag{3}$$

where $\delta_{1n} = 1$ if $n = 1$ and $\delta_{1n} = 1$ otherwise, C_n and S_n are Fourier's coefficients of $f(u, \dot{u})$. If we put $\tau = \Omega t$, we can see that C_n and S_n do not depend on Ω.
For $n = 0$, we have $q_0 = -\dfrac{C_0}{\omega_0^2}$
For $n = 1$, if the first harmonic amplitude is denoted by $\mathcal{A}^2 = p_1^2 + q_1^2$, equation (3) leads to following results

$$\Omega^2 = \omega_0^2 + \frac{(C_1 - F)q_1 + S_1 p_1}{\mathcal{A}^2} \tag{4}$$

The aforementioned equation is a relation between frequency Ω and the first harmonic amplitude $\mathcal{A}$. Therefore, we can use this equation to build a iteration schema where the amplitude $\mathcal{A}$ is fixed and the frequency Ω is recomputed by this relation after each iteration. The first harmonic (q_1, p_1) can be calculated from equation (3) in two ways which correspond to two iteration schemas.

$$p_1 = \frac{\Omega^2}{\omega_0^2} p_1 + \frac{\beta\Omega}{\omega_0^2} q_1 - \frac{S_1}{\omega_0^2} \quad \text{or} \quad p_1 = \frac{\omega_0^2}{\Omega^2} p_1 + \frac{\beta}{\Omega} q_1 + \frac{S_1}{\Omega^2} \tag{5}$$

For $n \geq 2$, we have

$$\begin{cases} q_n = \dfrac{\omega_0^2}{n^2\Omega^2} q_n + \dfrac{\beta}{n\Omega} p_n + \dfrac{C_n}{n^2\Omega^2} \\ p_n = \dfrac{\omega_0^2}{n^2\Omega^2} p_n - \dfrac{\beta}{n\Omega} q_n + \dfrac{S_{nk}}{n^2\Omega^2} \end{cases} \tag{6}$$

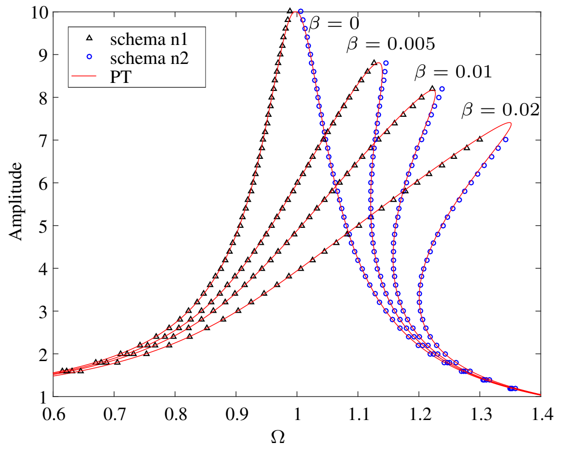
Applications

Duffing's oscillator

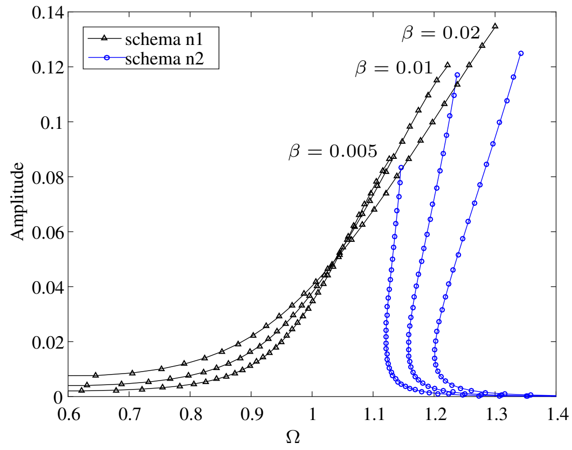
Let's consider a nonlinear oscillator given by equation (1) with $\omega = 1$, $\beta = 0.1$ and $F = 1$ and $f(u) = \varepsilon u^3$. The perturbation technique [1] gives an approximate of the periodic solution as follows

$$\left(\omega_0^2 - \Omega^2 + \frac{3}{4}\varepsilon\mathcal{A}^2\right)^2 + 4\beta^2\Omega^2 = \frac{F^2}{\mathcal{A}^2} \tag{7}$$

Figure 2a shows the periodic solution by the iteration procedure for different values of ε. We see that the new method converge well to all solutions of the oscillator. The calculation is performed with number of harmonic $N = 10$ and 100 iterations.



(a) First harmonic



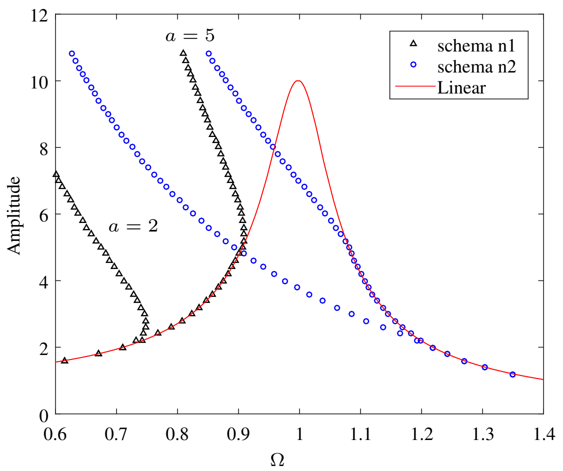
(b) Third harmonic

Figure 1: Forced Duffing oscillator by the perturbation technique (PT - red line) and the iteration method (markers)

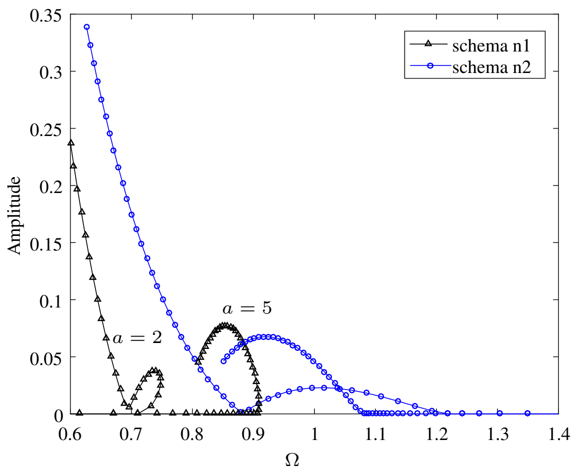
Elasto-elastic spring

When a spring-mass oscillator has an elasto-plastic behaviour, the nonlinear force is given by equation (8). Figure 2 show the results of the numerical method for difference elastic limit a with $k = 1$.

$$f(u) = \begin{cases} ku & \text{if } |u| \leq a \\ ka & \text{if } |u| > a \end{cases} \quad (8)$$



(a) First harmonic



(b) Third harmonic

Figure 2: Elasto-elastic spring by the iteration method for two different elastic limit a

The numerical applications show that new method has advantage to compute all solutions of a nonlinear oscillator including 'unstable' solutions. In addition, this approach with the frequency calculation give a new way to developed HBM for other dynamical problem.

References

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[3] T. Hoang, D. Duhamel, G. Foret, H. Yin, P. Argoul, Frequency dependent iteration method for forced nonlinear oscillators, Applied Mathematical Modelling 42 (2017) 441 – 448.