Global manifolds parametrised by isochrons

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Summary. Isochrons are sets of points in the basin of a stable periodic orbit that have the same asymptotic phase, that is, synchronise with a given point on the periodic orbit. We generalise this notion to periodic orbits of saddle type by considering the parametrisations of the stable manifold by forward-time isochrons and of the unstable manifold by backward-time isochrons. Computing these families of isochrons as (un)stable submanifolds of the period-map allows us to find, represent and illustrate two-dimensional global invariant manifolds in a new and efficient way.

Isochrons were introduced in 1974 by Winfree [9] as a means of viewing oscillatory dynamics in terms of a radial (contracting) component and a phase component. Any point in the basin of attraction of a periodic orbit Γ lies on precisely one isochron, namely, the one that corresponds to the point on Γ with which it will synchronise under the flow. Guckenheimer made the association with manifold theory [2] and showed that isochrons are manifolds of codimension one that are as smooth as the vector field itself; moreover, the collection of all isochrons of Γ foliate its basin. Hence, the dynamics of the system is entirely determined by the (discrete-time) dynamics on a single isochron. Winfree was keen to use this elegant theory in practice, but found that the computation of isochrons is rather challenging and, at the time, could only be done for relatively simple systems [5, 10]. Recently, there has been a renewed interest in isochrons and novel computational methods have been developed; we refer to [6] for a recent overview.
We compute the isochrons as one-dimensional parametrised curves with a method based on the continuation of suitable two-point boundary value problems [4, 8]. So far, this method has only been used in planar systems, for which the isochrons are one-dimensional manifolds. The approach can be used to compute isochrons of both attracting and repelling periodic orbits, in which case points on an isochron synchronise in backward time; we speak of forward-time and backward-time isochrons, respectively. Isochrons of a spiral sink or source equilibrium can also be defined and computed in a similar way [3, 6].

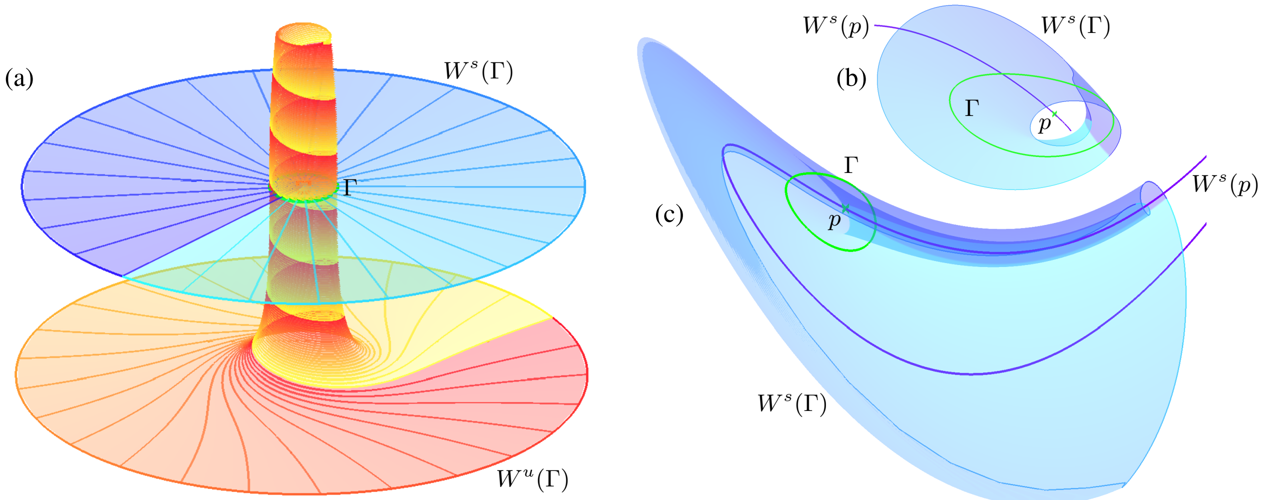

Figure 1: Parametrisation by isochrons of global manifolds of a periodic orbit Γ, where phase is indicated by colour shade. Panel (a) shows $W^s(\Gamma)$ and $W^u(\Gamma)$ for system (1); panels (b) and (c) show $W^s(\Gamma)$ of system (2) together with the one-dimensional stable manifold $W^s(p)$ of the saddle equilibrium p.

The key idea presented here is that isochrons foliate stable and unstable manifolds of saddle-type periodic orbits or equilibria in higher-dimensional systems. In particular, two-dimensional global manifolds can be computed as families of one-dimensional isochrons. We illustrate this idea by computing the respective forward- and backward-time isochrons of a saddle periodic orbit in the three-dimensional vector field

$$\dot{x} = \beta\, x - \omega\, y\,(1 - \kappa\, z) - x\,\frac{x^2 + y^2}{1 - (\zeta z)^3}, \quad \dot{y} = \omega\, x\,(1 - \kappa\, z) + \beta\, y - y\,\frac{x^2 + y^2}{1 - (\zeta z)^3}, \quad \text{and} \quad \dot{z} = \alpha\, z, \qquad (1)$$

which we designed in the same spirit as the example given in [9]. For $\alpha = 1.0$, $\beta = 1.0$, $\kappa = 0.15$, $\zeta = 0.125$, and $\omega = 2.0$, system (1) has a saddle periodic orbit Γ with two-dimensional stable and unstable manifolds, denoted $W^s(\Gamma)$ and $W^u(\Gamma)$, respectively. Figure 1(a) shows $W^s(\Gamma)$ and $W^u(\Gamma)$ calculated with the method from [5, 6] as one-parameter families of forward-time and backward-time isochrons, respectively. A selection from each family is shown as the curves on the surfaces, which are shaded according to the phase points on Γ; the discontinuity in the shading marks the location of the isochron of phase 0. Note that the isochron foliations of $W^s(\Gamma)$ and $W^u(\Gamma)$ highlight and illustrate how trajectories on the manifolds converge to Γ in forward or backward time.

Figures 1(b) and (c) show the two-dimensional stable manifold of a non-orientable periodic orbit, also denoted Γ, of the so-called ζ^3-model [1]

$$\dddot{x} + \ddot{x} + \beta\,\dot{x} + x\,(x - \alpha) = 0, \tag{2}$$

with $\alpha = 3.2$ and $\beta = 2.0$; we also plot the one-dimensional stable manifold $W^s(p)$ of a co-existing saddle equilibrium p. Figure 1(b) shows a first part of $W^s(\Gamma)$ that is approximately the size of this same manifold computed with a different method in [7, Figure 7]; note the half-twist in the surface, which is topologically a Möbius strip. Figure 1(c) shows a much larger part of $W^s(\Gamma)$ and provides a better illustration of the geometry that arises due to the non-orientability of Γ. As before, $W^s(\Gamma)$ is shaded according to the phase points on Γ and the discontinuity in shade marks the location of the isochron of phase 0. Observe how the colour shading bends with $W^s(p)$, indicating the enormous local stretching of the flow along one branch of $W^s(p)$.

The computation of a two-dimensional global stable or unstable manifold as a foliation of one-dimensional isochrons provides information about the dynamics on the manifolds that is complementary to the foliation by trajectories. This aspect is of particular interest for the study of manifolds of non-orientable periodic orbits and of interactions between global invariant manifolds.

References

[1] Arneodo, A., Coullet, P.H., Spiegel, E.A., Tresser, C. (1985) Asymptotic chaos. *Physica D* **14**:327–347.

[2] Guckenheimer, J. (1975) Isochrons and phaseless sets. *J. Math. Biol.* **1**:259–273.

[3] Hannam, J., Krauskopf B., Osinga, H.M. (2016) Global isochrons of a planar system near a phaseless set with saddle equilibria. *Eur. Phys. J. — Special Topics* **225**:2645–2654.

[4] Krauskopf B., Osinga, H.M. (2007) Computing invariant manifolds via the continuation of orbit segments. in *Numerical Continuation Methods for Dynamical Systems* Springer-Verlag, New York, pp. 117–154.

[5] Langfield, P., Krauskopf B., Osinga, H.M. (2014) Solving Winfree's puzzle: the isochrons in the FitzHugh-Nagumo model. *Chaos* **24**:013131.

[6] Langfield, P., Krauskopf B., Osinga, H.M. (2015) Forward-time and backward-time isochrons, and their interactions, *SIAM J. Appl. Dyn. Syst.* **14**:1418–1453.

[7] Osinga, H.M. (2003) Nonorientable manifolds of three-dimensional vector fields. *Int. J. Bifurcation & Chaos* **13**:553–570.

[8] Osinga, H.M., Moehlis, J. (2010) Continuation-based computation of global isochrons. *SIAM J. Appl. Dyn. Syst.* **9**:1201–1228.

[9] Winfree, A.T. (1974) Patterns of phase compromise in biological cycles. *J. Math. Biol.* **1**:73–93.

[10] Winfree, A.T. (2001) *The Geometry of Biological Time*, 2nd edition, Springer-Verlag, New York.