Stability of amplitude chimeras in oscillator networks

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Summary. We show that amplitude chimeras in ring networks of Stuart-Landau oscillators with symmetry-breaking nonlocal coupling represent saddle-states in the underlying phase space of the network. Chimera states are composed of coexisting spatial domains of coherent and of incoherent oscillations. We calculate the Floquet exponents and the corresponding eigenvectors in dependence upon the coupling strength and range, and discuss the implications for the phase space structure. The existence of at least one positive real part of the Floquet exponents indicates an unstable manifold in phase space, which explains the nature of these states as long-living transients. Additionally, we find a Stuart-Landau network of minimum size $N = 12$ exhibiting amplitude chimeras.

The dynamical state of networks of homogeneously coupled identical elements can show a peculiar behavior by self-organizing into two spatially separated domains with dramatically different behavior, e.g., a spatially coherent and a spatially incoherent region, which is called *chimera state* [1, 2]. Recently, a special type of chimera state has been discovered where coherence and incoherence occur with respect to the amplitude of the oscillators while all the elements of the network oscillate periodically with the same frequency and correlated phase [3, 4]; these are called *amplitude chimeras*. They emerge in networks with symmetry-breaking coupling.

The amplitude chimera is defined as the coexistence of two distinct domains separated in space: one subpopulation is oscillating with spatially coherent amplitude and the other exhibits oscillations with spatially incoherent amplitudes, i.e., the sequence of amplitudes of neighboring oscillators is random. In the incoherent domain the oscillation amplitudes and centers of mass are uncorrelated, whereas the phases of neighboring oscillators are correlated, in contrast to classical chimeras (e.g., phase chimeras and amplitude-mediated chimeras).

It is important to note that in the amplitude chimera state individual oscillators never become chaotic: they remain periodic and thus in the incoherent domain all the oscillators are temporally periodic but spatially chaotic. Amplitude chimeras are transients towards the completely synchronized state. Their lifetime strongly depends on the initial conditions. In contrast to classical chimeras [5, 6], where the transient time exponentially increases with the system size, for amplitude chimeras the transient time decreases and saturates for large system size [7].

In the present study we investigate the phase-space structure of this chimera pattern [8]. Amplitude chimeras are found to represent saddles in the high-dimensional phase space of the corresponding network, i.e., they have both stable and unstable manifolds, which explains their transient nature. Our goal is to investigate amplitude chimeras in a ring of Stuart-Landau oscillators with non-local coupling and to study their stability using Floquet analysis. In particular, we calculate the real parts of the Floquet exponents for a wide range of system parameters (coupling strength and range) in the regime where amplitude chimeras exist. Positive real parts of Floquet exponents correspond to the unstable manifold of the saddle and negative ones characterize its stable manifold. All investigations are performed for a deterministic system without noise and time-delay. Using the eigenvectors spanning the stable and unstable manifolds of the saddle cycle, we relate the amplitude chimera lifetime to the geometric structure of the phase space, and investigate the structural change with increasing coupling range, and its effect upon the lifetime.

We consider a ring network of N Stuart-Landau oscillators, $j \in \{1, ..., N\}$, all indices modulo N, which are coupled with the strength σ to their P nearest neighbors in each direction ($r = P/N$ defines the dimensionless coupling range):

$$\dot{z_j} = f(z_j) + \frac{\sigma}{2P} \sum_{k=j-P}^{j+P} (Re(z_k) - Re(z_j)), \tag{1}$$

where

$$f(z_j) = (\lambda + i\omega - |z_j|^2) z_j, \tag{2}$$

and $z_j = x_j + i\, y_j = r_j e^{i\phi_j} \in \mathbb{C}$, with $x_j, y_j, r_j, \phi_j \in \mathbb{R}$, and $\lambda, \omega > 0$. Without coupling the system undergoes a Hopf bifurcation at $\lambda = 0$, so that for $\lambda > 0$ a single Stuart-Landau oscillator performs self-sustained oscillations with frequency ω and follows the limit cycle trajectories with the radius $r_j = \sqrt{\lambda}$, and the unique fixed point ($x_j = y_j = 0$) is unstable. The coupled system (1) exhibits coherent in-phase synchronized oscillations or travelling waves, and partially coherent, partially incoherent amplitude chimera states, which are long-lasting transient states.

Amplitude chimera states have been found for small values of the coupling range r and sufficiently strong coupling strength. Decreasing the coupling strength σ for a fixed value of coupling range r corresponds to an increase of the lifetime of the amplitude chimera. The same tendency is observed for a fixed value of σ when increasing r. Thus, amplitude chimeras are particularly long-living in networks with weak coupling and large coupling range. Most amplitude chimera states have one positive real part Λ of the Floquet exponents, but some have two, corresponding to one or two unstable directions in phase space, respectively.

For a fixed coupling range r we observe an increase of the largest positive real part Λ of the Floquet exponents with increasing σ. Similarly, when changing the network topology by increasing r at fixed σ, we also observe that Λ increases. We find that within the same network topology, i.e., fixed r, the transient times decrease while the positive real part of the dominant Floquet exponent increases. The escape rate Λ from the saddle along the unstable direction increases, which leads to a shorter transient time $t_{tr} \sim \frac{1}{\Lambda}$. However, for constant σ with increasing r both the lifetime of the amplitude chimera and Λ increase.

Based upon the Floquet eigenvectors, we will now discuss the change in phase space structure induced by varying the network topology, i.e., the coupling parameter r. The geometric features of phase space in the vicinity of the amplitude chimera are such that trajectories approach the amplitude chimera saddle along the stable manifold spanned by the eigenvectors with negative real part of the Floquet exponent, and then escape along the unstable manifold spanned by the eigenvectors with positive real part, with escape rate Λ. If more neighbors are coupled to each element, i.e., for larger r, this leads to an increase in the width of the incoherent domain because those synchronized elements which are located at the edge of the coherent domain experience more influence from the elements in the incoherent domain, and hence they also become desynchronized. A larger incoherent domain means that it takes longer time to reach the completely synchronized global attractor, hence the lifetime increases. With increasing N the width for fixed r shrinks, which explains why the lifetime decreases with increasing N, in contrast to classical chimeras [5, 6].

These results verify the hypothesis [7] that amplitude chimeras are saddle-orbits in phase space with a small number (one or two) of unstable dimensions.

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