Folding tori and Chenciner bubbles in an ENSO model with delayed feedback

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Summary. We consider the sudden disappearance of stable invariant tori in a delay differential equation (DDE) model of the coupled El Niño Southern Oscillation (ENSO) system. This phenomenon can be explained by the folding of invariant tori and their resonance tongues, which are associated with bifurcation structures referred to as Chenciner bubbles. We identify and analyse a Chenciner bubble and show how tori of the DDE break up and disappear as invariant manifolds. These results can be interpreted as a novel type of tipping, induced by the presence of multiple-frequency dynamics.

An El Niño event refers to a much higher than average level of sea-surface temperature of the Pacific Ocean near the equatorial South American coast. Such events occur irregularly every 4–7 years, and they are known to arise from an ocean-atmosphere interaction that involves air pressure fluctuations above the central Pacific Ocean, called the Southern Oscillation. The overall ENSO system is of great concern because El Niño events have considerable impact across the globe on weather patterns, ecosystems and agriculture.

We are concerned here with the general class of conceptual models that consider the combined effects of seasonal forcing and different feedback loops. The latter arise from transport of salinity and/or temperature waves across the ocean, and they are modeled as delayed terms with a characteristic delay times. More specifically, we consider here the very simple model for ENSO introduced in [4], of the form

$$\dot{h}(t) = -b\tanh[\kappa h(t-\tau_n)] + c\cos(2\pi t), \tag{1}$$

where the variable h is the thermocline depth (a thin top layer of warmer water) at the easter Pacific Ocean, which is directly correlated to the sea-surface temperature. Time is measured in units of years and c is the strength of the annual periodic forcing. The negative feedback term has strength b and delay time τ_n, while κ represents the amount of coupling between atmosphere and ocean. Throughout, we fix $b = 1$ and $\kappa = 11$ and consider τ_n and c as bifurcation parameters. We remark that system (1) is a special case of a more complicated model introduced in [10], which also includes a positive feedback loop; see also [6] for more background information.

It has been shown that conceptual ENSO models with annual forcing and delayed feedback feature dynamics on invariant tori in large parameter regions, which are organised by resonance tongues where the dynamics is locked [6, 7, 8]. In particular, when resonance tongues overlap, one may find chaotic dynamics with certain characteristic features of ENSO. Our recent study [7] has identified a bistability between dynamics on different stable tori, in between what appear to be fold bifurcations of tori. We consider this phenomenon here in more detail. The motivation arises from the fact that fold bifurcations of smooth tori do not actually occur as such in generic systems. Rather, the tori need to break up by crossing so-called bubbles of further bifurcations that were first studied by Chenciner [2].

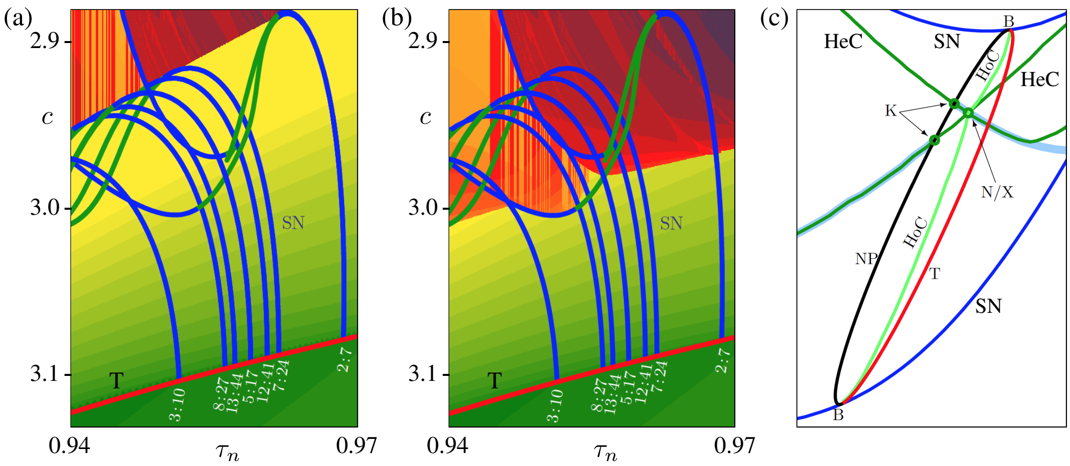

Figure 1: Folding resonance tongues in the (τ_n, c)-plane of (1) with colour indicating maxima of attracting solutions tracked when c is decreased (a) and increased (b); panel (c) shows the bifurcation structure near the lower fold of the $2:7$ resonance tongue.

Figure 1(a)–(b) shows $p:q$ resonance tongues in the (τ_n, c)-plane of (1) that emerge from a torus (or Neimark-Sacker) bifurcation curve T. More specifically, shown are the respective curves SN of saddle-node (or fold) bifurcations of locked periodic orbits that bound the resonance tongues, which themselves are rather narrow. The background colour indicates the maximum amplitude of stable solutions that were found when solutions were tracked by numerical simulation for decreasing and increasing c, respectively. The difference in colouring between panels (a) and (b) of Fig. 1, hence, indicates that there is a considerable region of bistability, which actually concerns two stable tori of different sizes [7].

Notice that the boundary of the region of bistability is not a single smooth curve, but rather appears to be formed by the envelopes of folding resonance tongues. The investigation of the region near one such fold, namely that of the $2:7$ resonance tongue, reveals a complicated structure. As Fig. 1(c) illustrates, it involves further curves of torus bifurcations T, homoclinic bifurcations HoC and heteroclinic bifurcations HeC, which are organised by points B, K and N/X of certain codimension-two bifurcations. This bifurcation diagram and associated phase portraits (not shown here) are in close agreement with the minimal unfolding of the dynamics in a Chenciner bubble that was suggested theoretically in [1]. To our best knowledge, our results represent the first identification and analysis of a Chenciner bubble in a system with delay. They have been made possible by advanced numerical methods for DDEs [3], including the computation of unstable manifolds in a suitable Poincaré section [5].
Some of the mathematical detail of the bifurcation structure we found may seem not be immediately relevant in the context of ENSO. However, Chenciner bubbles and the associated disappearance of a stable invariant torus can be interpreted as a novel, multi-frequency type of tipping. Generally, a (climate) tipping event refers to a situation where the small but steady variation of a parameter leads to a qualitatively different or drastic response [9]. The classic examples are saddle-node or fold bifurcations of equilibria and periodic orbits. Folding tori can be seen as the equivalent of this mechanism when several frequencies are involved in the dynamics. As we have seen, this phenomenon involves crossing Chenciner bubbles. It is an interesting subject for further study to determine possible precursors that indicate imminent tipping of this sort.

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