Stability and Control of the Fractional Damped Delayed Mathieu Equation

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Summary. Two novel methods are proposed to study the stability of linear fractional periodic time-delayed (FPTD) systems while the fractional damped delayed Mathieu equation is used as an illustrative example. The first method, called the explicit harmonic balance (EHB) method, yields sufficient conditions for fold, flip, and secondary Hopf transition curves in linear FPTD systems. The second method, called the fractional Chebyshev collocation (FCC) method, approximates the monodromy matrix for linear FPTD systems. The approximated monodromy matrix yields a linear period-to-period map. Transition curves of the fractional damped delayed Mathieu equation are first obtained by using the EHB method, and subsequently are compared with the stability regions obtained by the FCC method. Finally, a stabilizing control strategy using linear full state feedback is proposed for this system in which the control is allowed to employ periodic gains as well as fractional and delayed feedback of the state.

Introduction

Fractional order models (FOMs) have been used in modeling a variety of different physical systems, and their advantages compared to those of integer order models have been demonstrated, e.g. [1-5]. The stability analysis of a dynamical system is important in understanding how changes in system and design parameters cause a bounded or unbounded response and is an important key step in the design of a stabilizing feedback controller. In fractional differential equations (FDEs), the integer order of the derivative operator is relaxed to include fractional orders. These modifications bring additional complexity to stability analysis and control, which requires the development of new analytical and numerical strategies. Consequently, the stability and control of linear systems obtained from including these three phenomena (i.e. fractional order derivatives, periodic coefficients, and time delays) is the focus of this abstract, with a focus on the stability and control of the fractional damped delayed Mathieu equation.

The Fractional Damped Delayed Mathieu Equation

Consider the fractional damped delayed Mathieu equation of the form

$$\begin{aligned} &\ddot{x}(t)+\big(a+b\cos(\Omega t)\big)x(t)=c\,{}_0^C\mathcal{D}_t^\alpha x(t-\tau)+u(t) \\ &x(t)=\phi(t), \qquad -\tau\le t<0, \end{aligned} \tag{1}$$

where $\Omega=2\pi$, $\tau=1$ (for equal delay and parametric excitation period), and the left-sided Caputo derivative is defined as [1]

$${}_a^C\mathcal{D}_x^\alpha f(x)={}_a\mathcal{J}_x^{\lceil\alpha\rceil-\alpha}D^{\lceil\alpha\rceil}f(x) \tag{2}$$

where $a\in\mathbb{R}$, $\alpha\in[0,1]$, $\lceil x\rceil$ is the ceiling function, D is the integer derivative operator, and ${}_a\mathcal{J}_x^\alpha(.)$ is the left-sided fractional integral defined as ${}_a\mathcal{J}_x^\alpha f(x)=\dfrac{1}{\Gamma(\alpha)}\int_a^x f(\xi)(x-\xi)^{\alpha-1}d\xi$ in which $\Gamma(\cdot)$ denotes the Gamma function.

Transition Curves Using Explicit Harmonic Balance Method

The solution can be expressed in algebraic form by using operational matrices [2-3]. Furthermore, the Hill matrix of Eq. (1) is defined by linear operational matrices ($\mathcal{O}_{\mathbf{p}_N}(\cdot)$) as

$$\mathfrak{H}_{\mathbf{p}_N}:=\mathcal{O}_{\mathbf{p}_N}\big(A(t)\dot{x}(t)\big)+\mathcal{O}_{\mathbf{p}_N}\big(B(t)\,\mathcal{D}_t^\alpha x(t)\big)+\mathcal{O}_{\mathbf{p}_N}\big(C(t)\,\mathcal{D}_t^\beta x(t-\tau)\big) \tag{3}$$

where the solution can be expanded in a Fourier series and represented by a decomposed representation as $x(t)=\mathbf{p}_N^T\,I_{2N+1}\,\Theta_{\exp(\cdot)}^N(i\Omega t/2)$ in which $\mathbf{p}_N^T$ is the Fourier coefficient vector and I_{2N+1} is the identity matrix. The necessary and sufficient condition to have a solution of the assumed form is that the Hill matrix should be singular, which gives the flip and fold transition curves corresponding to Floquet multipliers equal to ±1. Moreover, for secondary Hopf instability, the Floquet multipliers are a complex conjugate pair with unit magnitude. Figure 1 shows the flip and fold transition curves of Eq. (1) in different parametric stability charts. In the $a-b$ parametric stability chart, increasing the coefficient c of the delay term alters the intersection of the transition curves on the a-axis.

Stability and Control Using the Fractional Chebyshev Collocation Method

The initial function $\phi(t)$ of system (2) in Banach space $X=\mathcal{C}\big([-\tau,0],\mathbb{R}^n\big)$ is denoted by $\mathcal{X}_0\in X$, i.e. $\mathcal{X}_0=\phi(\theta)$, $\theta\in[-\tau,0]$. Similarly, $\mathbf{x}(t)$ in the time interval $t\in\big[(j-1)\tau,j\tau\big]$ is represented by $\mathcal{X}_j\in X$. The steady-state monodromy operator of the linear periodic FDDE in Eq. (2) in Banach space X is defined as $\lim_{m\to\infty}\mathcal{X}_m=\lim_{m\to\infty}\mathcal{B}(\mathcal{X}_{m-1})$. By using the short-memory principle, the monodromy operator can be rewritten in terms of an approximation for this monodromy operator $\tilde{\mathcal{B}}: X \to X$ as $\mathcal{X}_j \approx \tilde{\mathcal{B}}^j(\mathcal{X}_0)$ where $\tilde{\mathcal{B}}^j(.) := \tilde{\mathcal{B}}(\tilde{\mathcal{B}}(\cdots\tilde{\mathcal{B}}(.)))$. Let $\tilde{\mathbf{B}}$ denote the discretized matrix approximation of operator $\tilde{\mathcal{B}}$ by using the fractional Chebyshev differentiation matrix [3-5]. Applying the induced norm $\|(\cdot)\|$ on the finite dimensional discretization of $\mathcal{X}_j \approx \tilde{\mathcal{B}}^j(\mathcal{X}_0)$, one can write $\|\mathbf{X}_j\| \le \|\tilde{\mathbf{B}}\|^j \|\mathbf{X}_0\| \le \rho^j \|\mathbf{X}_0\|$ where ρ is the spectral radius of the discretized monodromy matrix $\tilde{\mathbf{B}}$. The necessary and sufficient condition for the origin of Eq. (1) to be asymptotically stable is that all the eigenvalues of $\tilde{\mathbf{B}}$ lie inside the unit circle when the approximation error of using the short-memory principle is insignificant. This can be guaranteed by choosing N large enough [3]. The stability regions shown in Figure 2 in the $a-b$ parameter plane are obtained by using the FCC method when $u(t)=0$. Comparing this figure with Figure 1 reveals that the stability boundaries are a subset of the transition curves. Note that the $a=0.2$ and $b=0.1$ parameter set for the uncontrolled system corresponds to the unstable point denoted by a red solid circle in Figure 2.

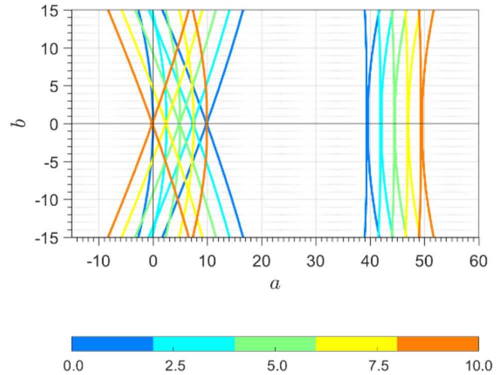

Figure 1: Flip and fold transition curves in *a*-*b* plane when $\alpha = 0$

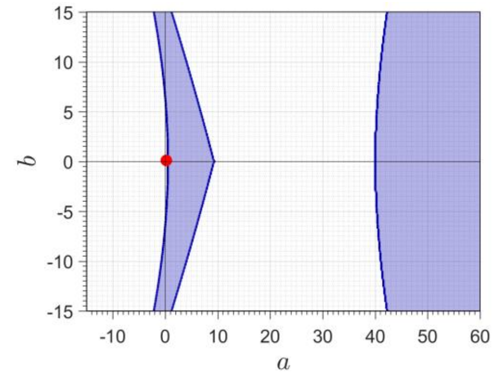

Figure 2: Flip and fold transition curves in *a*-*b* plane

Now consider a fractional delayed feedback control of the form

$$u(t) = k_{11}\, x(t-1) + k_{21}\, {}_0^C\mathcal{D}_t^\beta x(t-1) \tag{4}$$

where $k_{11}, k_{21} \in \mathbb{R}$, and $0 \le \beta \le 1$. The stability chart of the closed-loop system is shown in Figure 3 for different fractional orders. Consider changing the constant gains of the feedback control in Eq. (4) to periodic gains with two harmonics as

$$\begin{aligned} u(t) = &\left(k_{11} + k_{12}\cos(2\pi t) + k_{13}\sin(2\pi t)\right)x(t-1) + \\ &\left(k_{21} + k_{22}\sin(2\pi t) + k_{23}\cos(2\pi t)\right){}_0^C\mathcal{D}_t^\beta x(t-1) \end{aligned} \tag{5}$$

where $k_{ij} \in \mathbb{R}$, $i = 1,2$, $j = 1,2,3$. Figure 4 shows that for higher values of β adding more gain harmonics results in a lower spectral radius and hence a faster response.

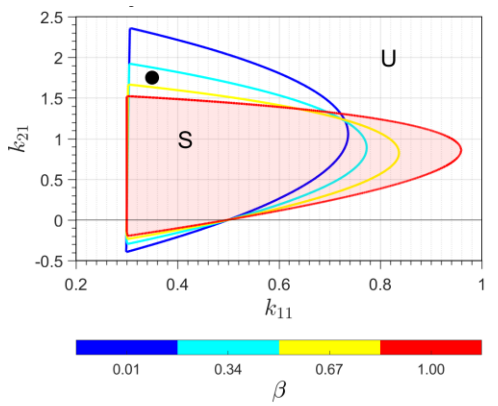

Figure 3: Stability chart for controlled system with the feedback control as Eq. (4) for a range of fractional order. S and U denote the stable and unstable regions, respectively.

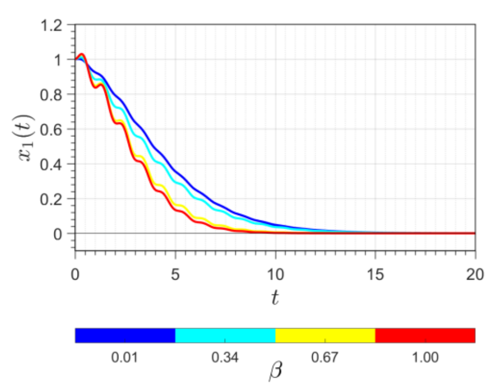

Figure 4: The response using the feedback control in Eq. (5)

Conclusion

Two techniques were introduced to examine the transition curves and stability of linear periodic fractional delay differential equations (FDDEs). The explicit harmonic balance (EHB) method was first introduced to find the transition curves by employing operational matrices associated with a Fourier basis. Conditions for the existence of nontrivial solution were obtained by setting the determinant of an infinite dimensional matrix, i.e. the Hill's matrix, to zero. Second, the fractional Chebyshev collocation (FCC) method was introduced to examine the stability of linear periodic FDDEs. By employing the short memory principle for fractional derivatives, the approximation of the monodromy operator was defined based on the state transition operator evaluated in the first period. The approximate solution was then obtained by discretizing the solution at Gauss-Lobatto-Chebyshev points. The advantages of the EHB and FCC methods were illustrated in studying the stability of the fractional damped delayed Mathieu equation with and without control.

References

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