Probabilistic Quantification of Extreme Events in Complex Systems

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Summary. We consider the problem of the probabilistic quantification of dynamical systems that have heavy-tailed characteristics. Here we develop a computational method, a probabilistic decomposition-synthesis technique, that takes into account the nature of internal instabilities to inexpensively determine the non-Gaussian probability density function for any arbitrary quantity of interest. Our approach relies on the decomposition of the statistics into a 'non-extreme core', typically Gaussian, and a heavy-tailed component. The proposed approach allows for the accurate quantification of non-Gaussian tails at more than 10 standard deviations, at a fraction of the cost associated with the direct Monte-Carlo simulations.

Introduction

Quantifying extreme or rare events is a central issue for many technological processes and natural phenomena. As extreme events, we consider rare transient responses that push the system away from its statistical steady state, which often lead to catastrophic consequences. Complex systems exhibiting rare events include (i) dynamical systems found in nature, such as the occurrence of rare climate events and turbulence, formation of freak water waves in the ocean; but also (ii) dynamical systems in engineering applications involving mechanical components subjected to stochastic loads, ship rolling and capsizing.

Description of the Method

Let the *dynamical system* of interest be governed by the following stochastic partial differential equation (SPDE):

$$\frac{\partial u(x,t)}{\partial t} = \mathcal{N}[u(x,t);\omega], \quad x \in D, \;\; t \in [0,T], \;\; \omega \in \Omega, \tag{1}$$

where $\mathcal{N}$ is a general (nonlinear) differential operator with appropriate boundary conditions. We assume that the system response is an ergodic process and that the system converges to a stationary probability measure. Here we are interested in determining the statistical distribution for a *quantity of interest* given by a functional of the solution $u(x,t)$ or as a solution of another dynamical system subjected to $u(x,t)$:

$$q = q[u(x,t)], \quad \text{or} \quad \frac{dq}{dt} = \mathcal{M}[q, u(x,t)]. \tag{2}$$

We assume that all rare event states, due to internal instabilities, defined by the condition $\|u\| > \zeta$, with ζ being the rare event threshold, 'live' in a low dimensional subspace V_s. We then decompose the response of the system as [1]:

$$u(x,t) = u_b(x,t) + u_r(x,t), \quad \text{with} \;\; u_r = \Pi_{V_s}[u], \quad \text{if} \;\; \|u\| > \zeta, \quad \text{and} \;\; u_b = u - u_r, \tag{3}$$

where Π_{V_s} denotes the linear projection to the subspace V_s. Above, u_r describes the evolution of the rare and extreme component of u in the subspace V_s provided the norm of the response satisfies the rare event threshold (i.e. this component describes transient events due to intermittent instabilities) and u_b is the background component that is given by the response excluding all rare responses, i.e. $u_b = u - u_r$. This conditional decomposition allows for the study of the two components separately (but taking into account mutual interactions), using *different* uncertainty quantification methods that (i) take into consideration the possibly high-dimensional (broad spectrum) character of the stochastic background, and (ii) the nonlinear and unstable character of rare events [1].
The application of this decomposition relies on the following assumptions [1]:

- A1 The existence of intermittent events have negligible effects on the statistical characteristics of the stochastic attractor and can be ignored when analyzing the background state u_b.
- A2 Rare events are statistically independent from each other.
- A3 Rare events are characterized by low-dimensional dynamics.

The analysis of the two regimes consists of the following steps [1]:

1. **Order-reduction** in the subspace V_s in order to model the rare event dynamics, expressed through u_r. Then using the approximation $u(x,t) \simeq u_r(x,t)$ we will compute the conditional pdf $\rho(q \mid \|u\| > \zeta, u_b \in R_e)$, under the condition that an extreme event occurs due to an internal instability in R_e.
2. **Quantification of the instability region** R_e using the reduced-order model, by analyzing the conditions that lead to a rare event.

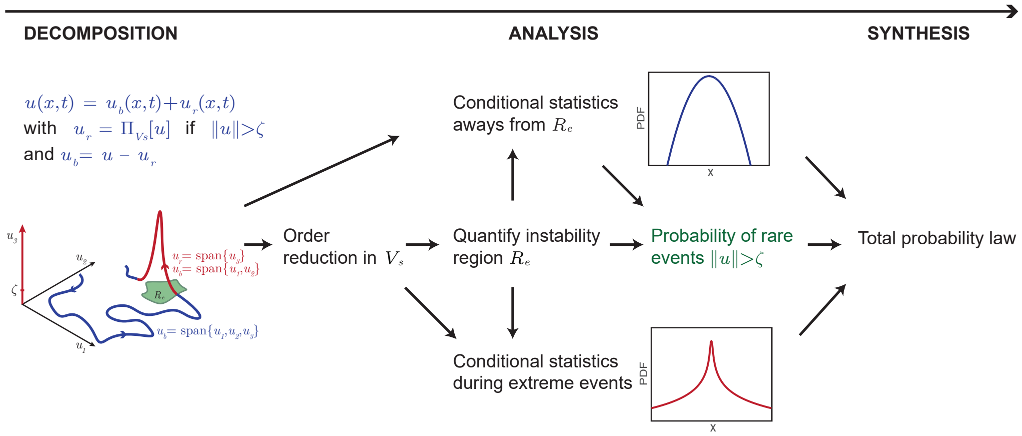

Figure 1: Outline of the steps of the decomposition-synthesis method.

3. **Description of the background dynamics**, expressed through the statistics of u_b, which is not influenced by any internal instabilities in R_e. Thus, when the response is dominated only by the background dynamics, we have $u(x,t) = u_b(x,t)$ and the pdf for the quantity of interest is given by $\rho(q \mid u = u_b)$.
4. **Probability for rare events due to internal instabilities** $\mathbb{P}(\|u\| > \zeta, u_b \in R_e)$, which quantifies the total time/space that the response spends in the rare event regime due to the occurrence of instabilities.

The next step of our technique is to probabilistically synthesize the information obtained from the previous analysis. Using a total probability argument, in the spirit of [2], we obtain the statistics for the quantity of interest q by

$$\rho(q) = \rho(q \mid \|u\| > \zeta, u_b \in R_e)\,\mathbb{P}_r + \rho(q \mid u = u_b)(1 - \mathbb{P}_r), \tag{4}$$

where $\mathbb{P}_r = \mathbb{P}(\|u\| > \zeta, u_b \in R_e)$ is the probability of a rare event due to an instability. The first term expresses the contribution of rare events due to internal instabilities and is the heavy-tailed part of the distribution for q. The second term expresses the contribution of the background state and accounts for the main probability mass in the pdf for q.
This total probability decomposition (4) separates the full response into the conditionally extreme response and the conditionally background response, weighted by their appropriate probabilities. The decomposition separates statistical quantities according to the total probability law through conditioning on *dynamical regimes*. In this manner, our approach connects the statistical quantities that we are interested in with important dynamical regimes that determine the dominant statistical features (e.g. a Gaussian core due to the background state and exponential like heavy-tails due to intermittent bursts). An outline of all the steps involved is presented in Figure 2.

Applications

We apply the method in two strongly nonlinear systems, one consisting of coupled nonlinear oscillators with intermittent energy exchanges and the other is the propagation of strongly nonlinear water waves described by the modified Nonlinear Schrodinger equation. The results and the comparison of the associated pdf are shown in Figure 2 (oscillators - left; nonlinear waves elevation - right).

References

[1] M. A. Mohamad, W. Cousins, and T. P. Sapsis. A probabilistic decomposition-synthesis method for the quantification of rare events due to internal instabilities. *Journal of Computational Physics*, 322:288–308, 2016.

[2] M. A. Mohamad and T. P. Sapsis. Probabilistic description of extreme events in intermittently unstable systems excited by correlated stochastic processes. *SIAM ASA J. of Uncertainty Quantification*, 3:709–736, 2015.

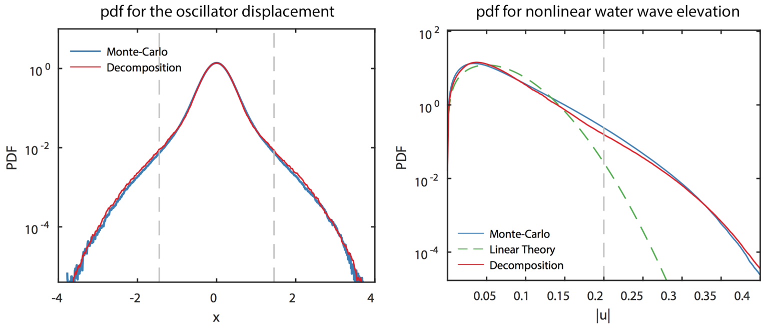

Figure 2: Comparison of the presented method with direct Monte-Carlo simulations - see [1] for details.