Energy method applied to the asymptotic methods of Non-linear mechanics

Hedrih (Stevanović) Katica*,**

*Department of Mechanics, Mathematical Institute of Serbian Academy of Science and Arts, Serbia; **Faculty of Mechanical Engineering at University of Niš, Serbia*

Summary. Modification of the Krilov-Bogolyubov-Mitropolyski asymptotic method of non-linear mechanics, and investigations of non-linear one- or multi-frequency free or forced, stationary or non-stationary oscillations of a class of the deformable bodies, is realized by the energy method. System of ordinary non-linear differential equations along amplitudes and phases of one or multi-frequency free, as well as forced stationary or non-stationary regimes, in first asymptotic approximation, are expressed by virtual work along deformable body point displacements. For the class of non-linear elastic homogeneous beams and thin elastic plates, system of ordinary non-linear differential equations along amplitudes and phases, in first asymptotic approximation are expressed. Interactions between non-linear modes in first asymptotic approximation are discussed.

Energy method applied to the Krilo-Bogolybov-Mitropolyski asymptotic method

For modification of the Krilov-Bogolyubov-Mitropolyski asymptotic methods [2, 3] of non-linear mechanics by energy method for investigation one frequency as well as multi-frequency regimes of transversal vibrations of non-linear elastic bodies, it is necessary to starts with known prepositions of asymptotic methods. Principal idea of the energy method applied to the Krilo-Bogolybov-Mitropolyski asymptotic method, is presented with partial differential equation of transversal vibration of the thin, non-linear elastic plate in the form:

$$\alpha^4 \frac{\partial^2 w(x,y,t)}{\partial t^2} + \nabla^4 w(x,y,t) = \varepsilon f\left(\tau, x, y, w(x,y,t), \frac{\partial w(x,y,t)}{\partial t}, \frac{\partial w(x,y,t)}{\partial x}, ..., \frac{\partial^4 w(x,y,t)}{\partial y^4}, \frac{\partial^5 w(x,y,t)}{\partial y^4 \partial t}, \vartheta_1(t), \vartheta_2(t), \vartheta_3(t)\right) \quad (1)$$

with boundary conditions in the form:

$$\mathfrak{L}_{ij}\left(w(x,y,t)\right)\Big|_{\substack{x=x_k\\ y=y_k}} \equiv \mathfrak{L}_{ij}\left(w(x,y,t), \frac{\partial w(x,y,t)}{\partial t}, \frac{\partial w(x,y,t)}{\partial x}, ..., \frac{\partial^4 w(x,y,t)}{\partial x^4}, \frac{\partial^4 w(x,y,t)}{\partial y^4}\right)\Bigg|_{\substack{x=x_k\\ y=y_k}} \equiv 0, \quad i,j,k = 1,2,3,4....M \quad (2)$$

and with initial conditions in the form

$$w(x,y,t)\big|_{t=0} = \sum_{k=1}^{k=3} p^{(k)} W_k(x,y) + \varepsilon^2 v_k(x,y) \approx \sum_{k=1}^{k=3} p^{(k)} W_k(x,y), \quad \frac{\partial w(x,y,t)}{\partial t}\bigg|_{t=0} = \sum_{k=1}^{k=3} q^{(k)} W_k(x,y) + \varepsilon^2 u_k(x,y) \approx \sum_{k=1}^{k=3} q^{(k)} W_k(x,y) \quad (3)$$

where $w(x,y,t)$ is transversal displacement of a thin plate, $N(x,y)$ is middle surface point, α^4 is coefficient presenting parameters of plate, ε is a small parameter, $\tau = \varepsilon t$ is slow changing time, $W_{sr}(x,y)\big|_{sr=1,2,3} \Rightarrow W_{k_s}(x,y)\big|_{k=1,2,3}$ are eigen amplitude functions of linear oscillations of corresponding plate satisfying boundary and orthogonality conditions, $p^{(k)}$ and $q^{(k)}$ some real parameters, $f\left(\tau, x, y, w(x,y,t), \frac{\partial w(x,y,t)}{\partial t}, \frac{\partial w(x,y,t)}{\partial x}, ..., \frac{\partial^4 w(x,y,t)}{\partial y^4}, \frac{\partial^5 w(x,y,t)}{\partial y^4 \partial t}, \vartheta_1(t), \vartheta_2(t), \vartheta_3(t)\right)$ is non-linear function of transversal displacement and its derivative, and periodic along $\vartheta_1(t), \vartheta_2(t), \vartheta_3(t)$ with periods of 2π, $\frac{d\vartheta_{sr}}{dt} = \nu_{sr}(\tau) \approx \omega_{sr}, \quad sr = 1,2,3$, ω_{sr} eigen circular frequencies of transversal linear vibrations of thin plate, and $\mathfrak{L}_{ij}$ is linear differential operators.

For three frequency vibration regime, we propose first asymptotic approximation of transversal displacements $w(x,y,t)$ of the plate middle surface points in the following form:

$$w(x,y,t) = \sum_{k=1}^{k=3} W_k(x,y) R_k(t) \cos\psi_k(t) + \varepsilon w_1(\tau, x, y, R_1, R_2, R_3, \psi_1, \psi_2, \psi_3, \vartheta_1, \vartheta_2, \vartheta_3) \quad (4)$$

where $\psi_k(t) = \vartheta_k(t) + \varphi_k(t)$ is full phases and $R_k(t)$ full amplitudes in first asymptotic approximation, defined by system of sixth non-linear differential equations of first order. Virtual work of the forces expressed by terms in right hand side of PDE (1) along virtual transversal displacements $w(x,y,t)$

$$\delta w(x,y,t) = \sum_{k=1}^{k=3} W_k(x,y)\left[\delta R_k(t)\cos\psi_k(t) - R_k(t)\sin\psi_k(t)\delta\psi_k(t)\right] + \varepsilon ...... \quad (5)$$

is in the following form [1, 2, 3]:

$$\delta\overline{\mathbf{W}} = \sum_{k=1}^{k=3} \sum_{p_1}\sum_{p_2}\sum_{p_3}\sum_{q_1}\sum_{q_2}\sum_{q_3}\left(\left(\frac{\delta\overline{\mathbf{W}}}{\delta R_k}\right)_{(p_1p_2p_3q_1q_2q_3)} \delta R_k + \left(\frac{\delta\overline{\mathbf{W}}}{\delta\varphi_k}\right)_{(p_1p_2p_3q_1q_2q_3)} \delta\varphi_k\right) = \frac{\varepsilon}{(2\pi)^6}\sum_{k=1}^{k=3}\delta R_k \sum_{p_1}\sum_{p_2}\sum_{p_3}\sum_{q_1}\sum_{q_2}\sum_{q_3} \mathbf{e}^{\mathbf{i}\sum_{j=1}^{j=3}(p_j\psi_j + q_j\vartheta_j)}$$

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\iint_A f_0(\tau, x, y, R_1, R_2, R_3, \psi_1, \psi_2, \psi_3, \vartheta_1, \vartheta_2, \vartheta_3) W_k(x,y)\cos\psi_k \mathbf{e}^{-\mathbf{i}\sum_{j=1}^{j=3}(p_j\psi_j + q_j\vartheta_j)} dA d\psi_1 d\psi_2 d\psi_3 d\vartheta_1 d\vartheta_2 d\vartheta_3 -$$

$$-\frac{\varepsilon}{(2\pi)^6}\sum_{k=1}^{k=3}R_k\delta\varphi_k\sum_{p_1}\sum_{p_2}\sum_{p_3}\sum_{q_1}\sum_{q_2}\sum_{q_3}\mathbf{e}^{\mathbf{i}\sum_{j=1}^{j=3}(p_j\psi_j+q_j\vartheta_j)} \tag{6}$$

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\iint_A f_0(\tau,x,y,R_1,R_2,R_3,\psi_1,\psi_2,\psi_3,\vartheta_1,\vartheta_2,\vartheta_3)W_k(x,y)\sin\psi_k\mathbf{e}^{-\mathbf{i}\sum_{j=1}^{j=3}(p_j\psi_j+q_j\vartheta_j)}dAd\psi_1d\psi_2d\psi_3d\vartheta_1d\vartheta_2d\vartheta_3$$

Ordinary non-linear first order differential equations along three amplitudes and there phases in first asymptotic approximation of first asymptotic approximation of solution in three frequency non-linear regime expressed by virtual work are in the following form:

$$\frac{dR_k}{dt}=\frac{2\varepsilon}{\alpha^4}\sum_{p_1p_2p_3}\sum_{q_1q_2q_3}\frac{\mathbf{i}\left\langle\sum_{j=1}^{j=3}p_j(\omega_j-\nu_j)\right\rangle\left(\frac{\delta\mathbf{W}}{\delta R_k}\right)_{p_1p_2p_3q_1q_2q_3}+2\omega_k\frac{1}{R_k}\left(\frac{\delta\mathbf{W}}{\delta\varphi_k}\right)_{p_1p_2p_3q_1q_2q_3}}{m_k\left[4\omega_k^2-\left\langle\sum_{j=1}^{j=3}p_j(\omega_j-\nu_j)\right\rangle^2\right]}$$

$$\frac{d\varphi_k}{dt}=\omega_k-\nu_k+\frac{2\varepsilon}{\alpha^4}\sum_{p_1p_2p_3}\sum_{q_1q_2q_3}\frac{\mathbf{i}\left\langle\sum_{j=1}^{j=3}p_j(\omega_j-\nu_j)\right\rangle\frac{1}{R_k}\left(\frac{\delta\mathbf{W}}{\delta\varphi_k}\right)_{p_1p_2p_3q_1q_2q_3}-2\omega_k\left(\frac{\delta\mathbf{W}}{\delta R_k}\right)_{p_1p_2p_3q_1q_2q_3}}{m_kR_k\left[4\omega_k^2-\left\langle\sum_{j=1}^{j=3}p_j(\omega_j-\nu_j)\right\rangle^2\right]},\ k=1,2,3 \tag{7}$$

An example of the application of the modified Krilo-Bogolybov-Mitropolyski asymptotic method

Let's consider tree-frequency vibration regime of thin elastic plate on non-linear elastic foundation and in linear damping, excited with distributed three frequency periodic force described by the following partial differential equation in the form:

$$\alpha^4\frac{\partial^2 w(x,y,t)}{\partial t^2}+\nabla^4 w(x,y,t)=-\varepsilon\beta[w(x,y,t)]^3+\varepsilon\overline{\mu}\frac{\partial w(x,y,t)}{\partial t}+\varepsilon\sum_{l=1}^{k=3}h_k(x,y,\tau)\sin\vartheta_k(t) \tag{8}$$

Firs asymptotic approximation of the solution for transversal displacement is in the form (4) and system of non-linear differential equation along three amplitudes and three phases in first asymptotic approximation using (6) and (7) is in the following form:

$$\frac{dR_{sr}}{dt}=-\frac{\varepsilon\mu R_{sr}}{4\alpha^4}-\frac{\varepsilon\mathbf{H}_{sr}(\tau)}{2(\omega_{sr}+\nu_{sr}(\tau))\alpha^4}\cos\varphi_{sr},\quad sr,mn,ab=1,2,3$$

$$\frac{d\varphi_{sr}}{dt}=(\omega_{sr}-\nu_{sr}(\tau))+\frac{3\varepsilon\beta}{16\omega_{sr}\alpha^4}\left\langle(R_{rs})^2\overline{\chi}(W_{rs})+2\left[(R_{mn})^2\overline{\chi}(W_{sr},W_{mn})+(R_{rcd})^2\overline{\chi}(W_{sr},W_{cd})\right]\right\rangle \tag{9}$$

$$+\frac{\varepsilon\mathbf{H}_{sr}(\tau)}{2R_{sr}(\omega_{sr}+\nu_{sr}(\tau))\alpha^4}\sin\varphi_{sr}$$

where

$$\mathbf{H}_{sr}(\tau)=\mathbf{H}_{sr}(h(x,y,\tau,),W_{sr}(x,y))=\frac{\iint_{(A)}h(x,y,\tau,)W_{sr}(x,y,)dA}{\iint_{(A)}[W_{sr}(x,y,)]^2dA} \tag{10}$$

Conclusions

Obtained system of non-linear differential equation along three amplitudes and three phased in first asymptotic approximation for three frequency oscillatory regimes using generalized form (6) and (7), and for special class in the form (9) obtained by energy method open large possibility for investigation non-linear phenomena in qualitative form as well in numerical experimentation of these system. From (9) we can analyze interactions between non-linear modes, appearance of the numerous resonant jumps, and also investigate internal resonances.

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