The impact of non-smoothness in the tyre-force characteristics on the nonlinear dynamics of towed vehicles

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Summary. In this study we investigate the lateral and yaw dynamics of towed road vehicles by means of an in-plane, single track model with an elastic tyre. It is shown how the dry friction in the tyre-ground contact can lead to piecewise-smooth continuous governing equations and how it influences the periodic orbits. The bifurcation of the rectilinear motion is thoroughly investigated using numerical collocation with the aim to point out the differences with respect to Hopf bifurcation in a smooth dynamical system. A further goal of the analysis is to discover the so-called bistable parameter ranges, where depending on the initial conditions the system can converge either to the rectilinear motion or a stable limit cycle. Additionally, a semi-analytical calculation is performed which can help to explore the structure of the nonlinear system in detail.

Introduction

Tyre deformation is one of the most important factors which influence the dynamics of road vehicle systems due to the fact, that the connection between the road and the vehicles is realised in the tyre-ground contact regions. One interesting feature of the contact is that the deformation, and as a result the achievable tyre forces are limited by dry-friction. Its effect is especially relevant in case of vibrations with larger amplitudes; therefore, taking the friction into account is essential in nonlinear analysis. Deriving the tyre-force characteristics based on a physical tyre model, such as the brush model typically leads to piecewise-smooth continuous functions for the lateral tyre force and the self aligning moment. The so-called semi-empirical tyre models (e.g. the widely used Magic Formula) neglect this effect by applying continuously differentiable functions [1]. This however, as shown in our study causes qualitative differences in the way the limit cycles develop from the linear stability boundary [2, 3, 4].

Mechanical model

Vehicle model

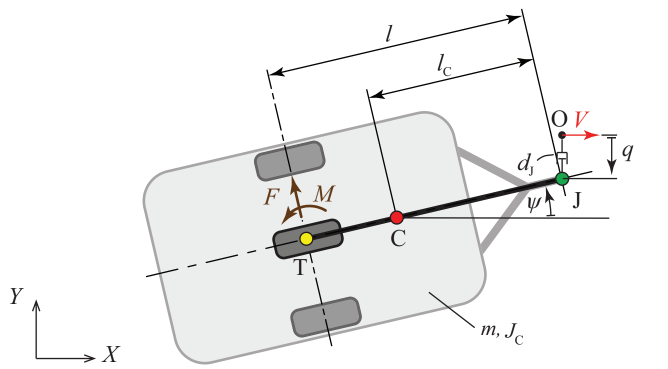

Figure 1: The in-plane single-track model of the towed trailer.

In our analysis we investigate a one-axle trailer towed with a constant velocity V. The towed vehicle is modelled by an in-plane single-track model as shown in Figure 1, by which we neglect the lateral extension. To obtain simple enough formulae for analytical studies the effect of the towing vehicle can be considered by a lateral spring and damper at the hitch point. However, as it can be shown, for large or medium speeds only the damping effect will be relevant in terms of the qualitative behaviour of the system. Thus, to simplify our formulae the lateral stiffness is neglected.

Non-smoothness in the tyre characteristics

The non-smoothness in the tyre force-characteristics is originated in the friction in the tyre-ground contact, which sets a limit for the tyre deformation and the transmittable tyre force. This can be well illustrated by means of the brush tyre model, considering a parabolic force-distribution as a limit for the side force [1]. This yields to a piecewise-smooth continuous function for the lateral force and self aligning moment in terms of the so-called side-slip angle, where the linear part is continuous, whereas the higher order terms feature discontinuities:

$$F(\alpha) = \begin{cases} F_{\max}, & \alpha < -\alpha_{\text{crit}}, \\ f_3\alpha^3 + f_2\alpha^2 + f_1\alpha, & -\alpha_{\text{crit}} \leq \alpha < 0, \\ f_3\alpha^3 - f_2\alpha^2 + f_1\alpha, & 0 \leq \alpha < \alpha_{\text{crit}}, \\ -F_{\max}, & \alpha_{\text{crit}} < \alpha \end{cases} \tag{1}$$

where α is the side-slip angle, $f_r,\ r = 1, 2, 3$ denote the r-th order term of the force-characteristics, whereas $F_{\max}$ is the upper limit of the side-force. Using the brush model the side-slip angle is calculated based on the vehicle kinematics, but one can also formulate a first order differential equation and consider some of the dynamic features of the tyre-ground contact.

Bifurcation analysis

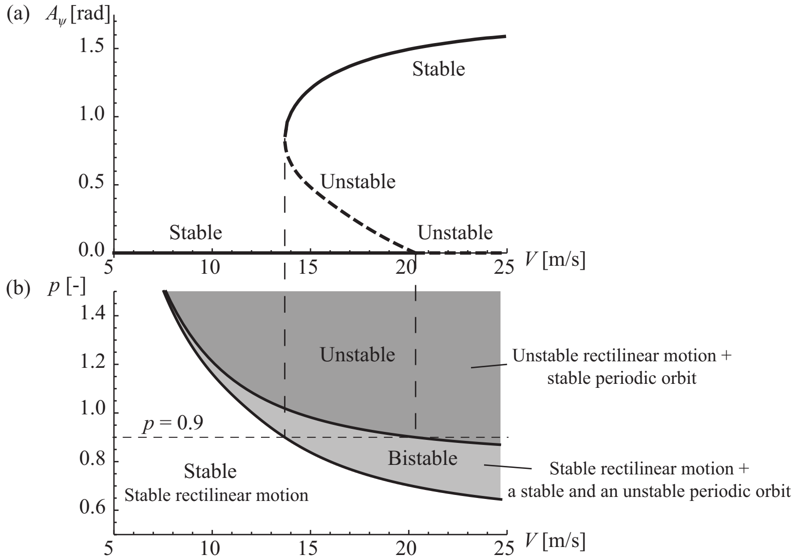

Figure 2: Bifurcation diagram showing the amplitude of the solutions (a); stability chart of the rectilinear motion by means of the towing speed V and the payload position $p = l_C/l$ (b).

The limit cycles in the system are calculated using the method of numerical collocation. The continuation on the branch of the periodic solutions is performed in Auto07p software [5]. Additionally, the linear stability boundary of the rectilinear motion as well as the so-called fold point (the location of saddle-node bifurcation of the limit cycles) is also calculated with respect to the velocity and the payload position (see Figure 2). Thus, a bistable zone can be obtained, where although the rectilinear motion is stable, a stable limit cycle exists, too. This makes important to identify the otherwise 'invisible' unstable limit cycle, which as a separatrix determines the magnitude of perturbation that makes the system diverge from the straight-line motion.

The results from the numerical collocation are compared with a semi-analytical method based on the Galerkin-technique. First, the centre manifold reduction has to be performed. [2] As in this case the non-smooth second order terms are relevant, for the centre manifold a linear approximation is required. Then, using the second order terms we can determine the branch of unstable limit cycles close to the linear stability boundary. Although this calculation gives an approximate result only for the periodic orbits, it reveals certain topological features of the bifurcation which are typical with piecewise-smooth systems, such as the branch of limit cycles emerges in a conical structure instead of a paraboloid [4]. The method gives an exact result for the stability of the rectilinear motion at linear stability boundary. Thus, it is capable to determine whether the bifurcation is sub- or supercritical, i.e. if a bistable zone can exist at the given parameter domain. A further aim is to extend this analysis by including the third order terms as well, that would make an analytical approximation possible for the location of the fold point and the limits of the bistable zone, too.

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References

[1] Pacejka H. B. (2002) Tyre and Vehicle Dynamics, Elsevier Butterworth-Heinemann, Oxford

[2] Kuznetsov Y. A. (1998) Elements of Applied Bifurcation Theory, Springer-Verlag, New York

[3] Simpson D. J. W. (2010) Bifurcations in piecewise-smooth continuous systems. World Scientific Publishing Co. Pte. Ltd., Singapore

[4] Leine R. I. (2006) Bifurcations of equilibria in non-smooth continuous systems. *Physica D*, 121-137.

[5] Doedel E. J., Oldeman B. E. (2012) Auto07p: Continuation and bifurcation software for ordinary differential equations, Concordia University, Montreal