Dynamics of a strongly non-linear mechanical system: a case of dissipation-induced instability

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Summary. Our goal in this work is to investigate existence, stability and bifurcations of periodic orbits in a strongly non-linear non-ideal problem. We have rigorously obtained existence of periodic orbits as well as a couple of inequalities which governs their stability. Moreover, in a special case turns out that the increasing of dissipation leads to instability of the periodic orbits. Such phenomenon has been reported in the literature mainly for linear systems.This is known as dissipation-induced instability.

Introduction

In the literature on non-ideal problems, see for example [1], only weakly non-linear problems are approached. A rigorous approach of the dynamics of a weakly non-linear non-ideal problem was performed in [3]. In that paper existence, stability and bifurcations of periodic orbits, which leads to Sommerfeld Effect, were investigated. Our goal in this work is to investigate the same questions in a strongly non-linear non-ideal problem. We have rigorously obtained existence of periodic orbits as well as a couple of inequalities which governs their stability. In a quite particular case of our results, in fact in an ideal problem, it was proved that the increasing of dissipation leads to instability of the periodic orbits. Such phenomenon has been reported in the literature mainly for linear systems, see [4]. This is known as dissipation-induced instability. It was necessary to do massive numerical and symbolic computations, which were performed by the CAS `Maxima`, `http://maxima.sourceforge.net/`.

The Centrifugal Vibrator

We consider a mechanical system excited by a *DC* motor with limited supply power, which base is supported on a spring. Besides, the *DC* motor rotates a small mass m, Figure 1. This mechanism is known as centrifugal vibrator. The

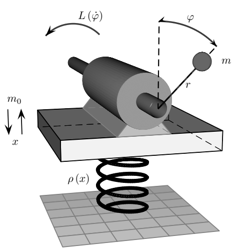

Figure 1: Centrifugal Vibrator

mathematical model is given by the following system:

$$\left\{\begin{array}{rcl} m_1\ddot{x}+\beta\dot{x}+cx+dx^3 &=& mr\dot{\varphi}^2\cos\varphi+mr\ddot{\varphi}\sin\varphi, \\ I\ddot{\varphi} &=& M(\dot{\varphi})+mr\ddot{x}\sin\varphi+mgr\sin\varphi. \end{array}\right. \tag{1}$$

For details of this model see [3]. The function $M(\cdot)$ is the difference between the driving torque of the source of energy (motor) $L(\dot{\varphi})$ and the resistive torque applied to the rotor. Such function $M(\cdot)$ is obtained from experiments. We can rewrite the equations of motion (1) as a system of first order. Take $\omega^2=\frac{c}{m_1}$, $a_2=-\frac{\beta}{m_1}$, $a_3=\frac{d}{m_1}$, $a_4=\frac{mr}{m_1}$, $a_5=\frac{mr}{I}$, $a_6=\frac{mgr}{I}$, $M_1(\dot{\varphi})=\frac{M(\dot{\varphi})}{I}$. Now, let us introduce a small parameter ϵ in these parameters. Let us replace the parameters a_i, $i\neq 3$ and ω by $\epsilon\, a_i$, $i\neq 3$ and $\epsilon\,\omega$ respectively. Moreover, let us substitute $M_1(\dot{\varphi})$ by $\epsilon\, M_1(\dot{\varphi})$. Then by making $x_1=x$, $x_2=\dot{x}$, $x_3=\varphi$, $x_4=\dot{\varphi}$ in (1), it is obtained a fourth order system which unperturbed part involves the term $a_3x_1^3$. From now on jacobian elliptic functions will be used in the next steps. The jacobian cosine with modulus $1/\sqrt{2}$ is denoted by $cn\left(t,1/\sqrt{2}\right)$ and its period by k_0. By taking $x_1=C\,cn\left(D,\frac{1}{\sqrt{2}}\right)$, $x_2=\sqrt{a_3}\,C^2\,cn'\left(D,\frac{1}{\sqrt{2}}\right)$ in this fourth order sytem, applying the usual reduction process in the obtained equation and using the following change of variables $D\left(x_3\right)=D_1\left(x_3\right)-\frac{k_0\,x_3}{2\,\pi}$ in the reduced system,one obtains

$$\begin{pmatrix} \frac{dC}{dx_3} \\ \frac{dD_1}{dx_3} \\ \frac{dx_4}{dx_3} \end{pmatrix} = \begin{pmatrix} 0 \\ u_{02} \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} \frac{\left(2\,\sqrt{a_3}\,a_4\,{x_4}^2\,\cos(x_3)\,cn'\left(D_1-\frac{k_0\,x_3}{2\,\pi},\frac{1}{\sqrt{2}}\right)+a_2\,C^2\left(a_3-a_3\,cn^4\left(D_1-\frac{k_0\,x_3}{2\,\pi},\frac{1}{\sqrt{2}}\right)\right)\right)}{2\,a_3\,x_4\,C} \\ -\frac{\left(a_2\,a_3\,C^2\,cn\left(D_1-\frac{k_0\,x_3}{2\,\pi},\frac{1}{\sqrt{2}}\right)cn'\left(D_1-\frac{k_0\,x_3}{2\,\pi},\frac{1}{\sqrt{2}}\right)+\sqrt{a_3}\,a_4\,{x_4}^2\,\cos(x_3)\,cn\left(D,\frac{1}{\sqrt{2}}\right)\right)}{a_3\,x_4\,C^2} \\ -\frac{a_3\,a_5\,C^3\sin(x_3)\,cn^3\left(D_1-\frac{k_0\,x_3}{2\,\pi},\frac{1}{\sqrt{2}}\right)}{x_4}+\frac{(M_1(x_4)+a_6\sin(x_3))}{x_4} \end{pmatrix} + O\left(\epsilon^2\right) \tag{2}$$

where $u_{02}=u_{02}\left(C,x_4\right)=\frac{2\,\pi\,\sqrt{a_3}\,C+k_0\,x_4}{2\,\pi\,x_4}$.

Existence of Periodic Orbits

Our theorem of existence is the following one. Let us assume that the following conditions are valid. The next inequality holds and there is $\left(\overline{a}, \overline{b}, \overline{c}\right)$ such that

$$\left|\frac{k_0^3\, a_2\, \left(k_6 - k_0\right) \cosh\left(\frac{\pi}{2}\right)}{2^{\frac{11}{2}}\, \pi^4 \sqrt{a_3}\, a_4}\right| < 1, M_1\left(-\frac{2\,\pi\,\sqrt{a_3}\,\overline{a}}{k_0}\right) = -\frac{k_0^2\, k_1\, a_2\, \left(k_6 - k_0\right)\sqrt{a_3}\, a_5 \cosh\left(\frac{\pi}{2}\right)\overline{a}^3}{2^{\frac{11}{2}}\, \pi^4\, a_4}, \tag{3}$$

$$\sin\left(\frac{2\,\pi\,\overline{b}}{k_0}\right) = -\frac{k_0^3\, a_2\, \left(k_6 - k_0\right) \cosh\left(\frac{\pi}{2}\right)}{2^{\frac{11}{2}}\, \pi^4 \sqrt{a_3}\, a_4}, \overline{c} = \frac{2\,\pi\,\overline{a}\,\sqrt{a_3}\, F\left(2\pi, \overline{a}, \overline{b}\right)}{k_0^2}, \tag{4}$$

$$2^{\frac{13}{2}}\, \pi^5\, a_4\, M_1'\left(-\frac{2\,\pi\,\sqrt{a_3}\,\overline{a}}{k_0}\right) - 3\, k_0^3\, k_8\, a_2\, \left(k_6 - k_0\right) a_5 \cosh\left(\frac{\pi}{2}\right)\overline{a}^2 \neq 0, \tag{5}$$

where $k_6 = \int_0^{k_0} cn^4\left(u, \frac{1}{\sqrt{2}}\right) du$, $k_8 = \int_0^{k_0} cn^3\left(u, \frac{1}{\sqrt{2}}\right)\cos\left(\frac{2\,\pi\,u}{k_0}\right) du$ and $F\left(s, a, b\right)$ has a huge expression and it will be omitted here. Then (2) has 2π periodic solutions for all ϵ adequately small. This result is rigorously proved by using the Regular Perturbation Theory and the Poincaré Method. A good reference is [2].

Stability

Let $h\left(s, \epsilon\right)$ be the 2π periodic solution obtained in the foregoing section. By using the Regular Perturbation Theory one obtains $h\left(s, \epsilon\right) = h_0\left(s\right) + h_1\left(s\right)\epsilon + O\left(\epsilon^2\right)$ where h_0, h_1 as well as the remainder $O\left(\epsilon^2\right)$ are 2π periodic mappings which are explicitly computed. The symbolic computations involved in are really big ones. The linearization of (2) at h yields a 2π periodic time dependent linear system $y'\left(s\right) = A\left(s, \epsilon\right) y\left(s\right)$, where A is a 3×3 matrix. Let us denote the principal matrix of this system by $N\left(s, \epsilon\right)$ where $N\left(0, \epsilon\right) = \text{identity matrix}$. The monodromy matrix is given by $\overline{N}\left(\epsilon\right) = M\left(2\pi, \epsilon\right)$. Again, by using the Regular Perturbation Theory in this system, one obtains $N\left(s, \epsilon\right) = N_0\left(s\right) + N_1\left(s\right)\epsilon + O\left(\epsilon^2\right)$. The symbolic computations involved in the obtaining of N_0, N_1 are big ones too. Consider $p\left(\epsilon, z\right)$ the characteristic polynomial of $\overline{N}$. It can be proved that $\frac{p\left(\epsilon\, z+1, \epsilon^2\right)}{\epsilon^3} = z^3 + c_2\left(\epsilon\right) z^2 + c_1\left(\epsilon\right) z + c_1\left(\epsilon\right)$. The Implicit Function Theorem can be applied in the right hand side of the last equation. One obtains that the roots r_1, r_2, r_3 of p are given by $r_1\left(\epsilon\right) = 1 + d_1\epsilon + O\left(\epsilon^{\frac{3}{2}}\right)$, $r_2\left(\epsilon\right) = 1 + i\, e_1\sqrt{\epsilon} + e_2\epsilon + O\left(\epsilon^{\frac{3}{2}}\right)$, where e_1, e_2 are real numbers. And, $r_3\left(\epsilon\right)$ is the complex conjugate of $r_2\left(\epsilon\right)$. All coeficients are explicitly computed. If $d_1 < 0$ and $2e_2 + e_1^2 < 0$ then $\left|r_1\left(\epsilon\right)\right| < 1$ and $\left|r_2\left(\epsilon\right)\right| < 1$ for $\epsilon << 1$. It is well known this leads to the stability of the periodic orbit. Otherwise, if $d_1 > 0$ or $2e_2 + e_1^2 > 0$ the the periodic orbit is unstable.

Dissipation-induced instability in the ideal case

Now let us to investigate a quite special case when $M_1\left(\cdot\right) = C_0$, C_0 constant. This leads to an ideal mechanical system. In view of (3) there is an upper bound for the dissipation $\left|a_2\right|$ which will be denoted by σ. From the results in the last section, it can be proved if the parameters a_3, a_4, a_5 are fixed with $1 < a_3 < 3$, take the applied torque C_0 adequately big then if $\left|a_2\right|$ is small then the periodic orbit is stable. If $\left|a_2\right|$ is near σ then the periodic orbit is unstable. This means the increasing of dissipation leads to an unstable periodic orbit. Such kind of phenomenon is known in the literature as dissipation-induced instability, for details see [4]. It must be emphazised the examples in given in [4] are linear ones.

Conclusions and Acknowledgements

By using a special change of variables, involving elliptic functions, in a strongly non-linear non-ideal problem questions on existence, stability and bifurcation can be efficiently approached. Particularly, the stability of the orbits are controlled by two inequalities. In a particular case, which is an ideal problem,it was found out the phenomenon of dissipation-induced instability. The next steps will be to investigate the ocurrence of such phenomenon, as well as,of the Sommerfeld Effect in other strongly non-linear systems.

The author acknowledges the support given by FAPEMIG and CNPq.

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