Twin canards and MMOs in a chemical reaction model

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Summary. We consider a system of three ordinary differential equations for autocatalytic reaction dynamics with one fast and two slow variables. Mixed-mode oscillation (MMO) consisting of alternating small- and large-amplitude oscillations are a prominent feature of the observed behaviour. We focus here on a parameter regime of MMOs where the time-scale-separation parameter is relatively large and compute the attracting and repelling slow manifolds and canard orbits as their intersections. We find that canard orbits appear in pairs — referered to as twin canards — that organise the slow manifolds into a structure of ribbons with different numbers of small-amplitude oscillations.

We consider the model for an autocatalytic reaction in a stirred tank [3, 4], which can be written in dimensionless form as

$$\dot{a} = \epsilon(\mu(\kappa + c) - ab^2 - a), \quad \dot{b} = ab^2 + a - b, \quad \text{and} \quad \dot{c} = \epsilon(b - c), \tag{1}$$

where a, b and c are (positive) concentrations of reactants. The parameter $\mu > 0$ represents the so-called pool concentration, κ is the constant rate of the initiation reaction and ϵ is the time-scale ratio. Throughout, we fix $\kappa = 2.5$ and consider ϵ and μ as bifurcation parameters.

A one-parameter bifurcation analysis in μ identifies many isolas of mixed-mode oscillations with different signatures of small and large amplitudes [3]. To gain insight into this structure, we compute the two-dimensional attracting and repelling slow manifolds and their intersections in canard orbits; see [1, 2] for details on the numerical set-up. For the representative example of $\mu = 0.295$ we find 68 canard orbits, which is a much larger number than the 28 that are predicted by the presence of a folded node in the singular limit. Hence, ϵ is in an intermediate regime beyond the singular limit [1]. We find that additional canard orbits are created in pairs in fold bifurcations when ϵ is changed. We refer to such a pair as twin canard orbits, because they bound ribbons on the attracting slow manifold S^a_ϵ with the same number of small-amplitude oscillations. We remark that, as such, twin canards are different from the canard orbits with folds considered in [2], which feature at least one large oscillation at or very near their folds in ϵ.

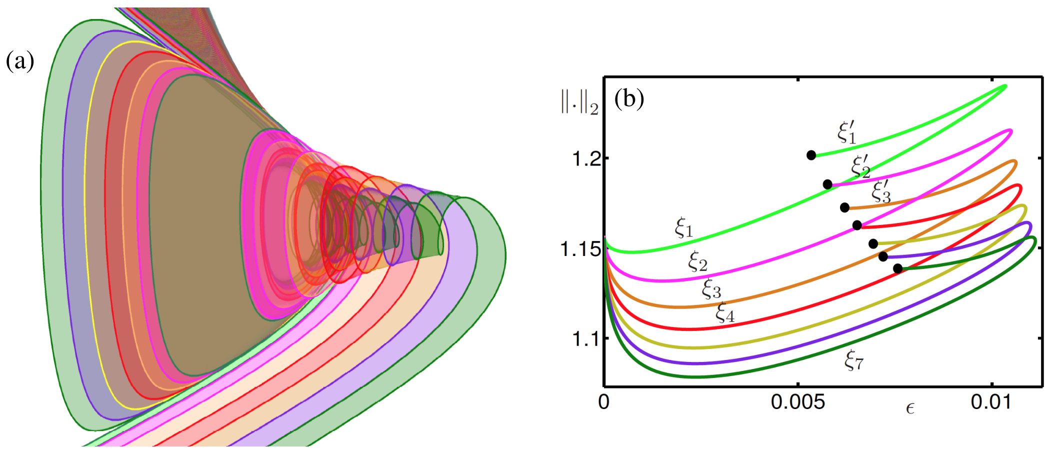

Figure 1: Ribbons bounded by twin canards of (1) for $\epsilon = 0.01$ and $\mu = 0.295$ (a), and continuation of canard orbits in ϵ (b).

Figure 1(a) shows seven ribbons for $\epsilon = 0.01$, which are bounded by twin canards ξ_i and ξ_i' for $i = 1, \ldots, 7$. As Fig. 1(b) shows, the twin canards are born in fold bifurcations; such a fold corresponds to a quadratic tangency between the two surfaces S^a_ϵ and S^r_ϵ. More specifically, the canards ξ_1–ξ_7 for $\epsilon = 0.01$ can be continued all the way to the singular limit $\epsilon = 0$. For increasing ϵ, they can be continued to their folds, where the continuation continues, for decreasing ϵ, to follow their twins ξ_1'–ξ_7'. We stop each continuation when one of the small-amplitude oscillations becomes too large (in a well-defined way), meaning that the orbit segment ceases to be the twin of the respective canard orbit.

We find more than 35 main ribbons, each bounded by a pair of twin canard orbits. In between every two neighbouring main ribbons there are further, secondary ribbons and associated twin canards. In this way, we obtain a comprehensive picture of the overall geometry of mixed-mode oscillations for larger values of ϵ, beyond the regime where the structure of canard orbits can be explained by the theory of folded nodes [1].

References

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