Damped Hill's Equation and Its Application to Attenuate Vibrations

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Summary. We define a class of linear Hamiltonian systems with dissipation whereby we develop the properties of μ-Symplectic and γ-Hamiltonian matrices. Due to the symmetry property, we can reduce the $2n$ dimensional to the n-dimensional linear system. The damped Hill's equation with n DOF is written as the new class of system, then we prove that monodromy matrix associated to the system is μ-Symplectic. With this new tool, we describe a method to attenuate vibration in a mechanical system with 2-DOF by parametric excitation.

Introduction

In mechanical systems there is always a small amount of dissipation; however, the most classical works, [1, 2] do not consider the linear Hamiltonian systems with dissipation, from now on we name them γ-Hamiltonian systems. It is also a well-known fact, that it is possible to increase the dissipation in a mechanical system by introducing parametric excitation [3], regarding control theory, this is understood as an open loop control technique eliminating the expensive measurements of the state. Through the averaging and perturbations methods is concluded that the maximal attenuation occurs close to the critical frequencies [3, 4]; however, these results are correct only for small parameters. Using the properties of the γ-Hamiltonian systems we can obtain the conditions for the attenuation occurrences for any parameter, eliminating the restrictive condition of small parameters approximation.

In the first and second section of this paper are written the properties of the μ-Symplectic and γ-Hamiltonian matrices. Then we prove that for any γ-Hamiltonian system, the transition state matrix is μ-Symplectic. The damped Hill's equation is described as γ-Hamiltonian periodic system and its monodromy matrix as μ-Symplectic. The Floquet factorization make possible to transform the γ-Hamiltonian periodic system into time-invariant γ-Hamiltonian system. Finally, the presented method makes possible to find the relations where the attenuation of mechanical systems occurs, the results are presented for a mechanical system with 2-DOF.

γ-Hamiltonian Matrices

Let be

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

an real square matrix where I_n is the $n \times n$ identity matrix, it is clear that $J^T = -J = J^{-1}$, $J^2 = -I_{2n}$, $\det(J) = 1$.

Definition 1 *The matrix $\tilde{A} \in \mathbb{R}^{2n\times 2n}$ is said Hamiltonian if and only if:*

$$\tilde{A}^T J + J\tilde{A} = 0$$

Let be $P_{\tilde{A}}(s)$ the characteristic polynomial of $\tilde{A}$, then $P_{\tilde{A}}(s)$, is an even polynomial, that is to say, it only has even powers, thus the eigenvalues of $\tilde{A}$ are symmetric with respect to the imaginary axis i.e. if s is an eigenvalue of $\tilde{A}$, $s \in \sigma\left(\tilde{A}\right)$ then $-s$ is an eigenvalue too $-s \in \sigma\left(\tilde{A}\right)$. Since the matrix $\tilde{A}$ is real, $\bar{s}$ and $-\bar{s}$ are eigenvalues as well. Then the eigenvalues of the Hamiltonian matrix are placed symmetrically with respect to both real and imaginary axis, thus the eigenvalues: appear in real pairs, purely imaginary pairs or complex quadruples [5, 1].

Definition 2 *The matrix $A \in \mathbb{R}^{2n\times 2n}$ is $\gamma-$Hamiltonian if and only if for some $\gamma \leq 0$*

$$A^T J + JA = 2\gamma J \tag{1}$$

From (1) it is clear that $(A + \gamma I_{2n})^T J + J(A + \gamma I_{2n}) = 0$ then A is γ-Hamiltonian if and only if $(A + \gamma I_{2n})$ is a Hamiltonian matrix. Then if $s + \gamma \in \sigma(A) \Rightarrow s \in \sigma(A + \gamma I_{2n})$, but $(A + \gamma I_{2n})$ is an Hamiltonian matrix, then $-s \in \sigma(A + \gamma I_{2n}) \Rightarrow -s + \gamma \in \sigma(A)$, the following lemma states this idea.

Lemma 3 *If $\gamma + s$ is an eigenvalue of A γ-Hamiltonian matrix then also $\gamma - s$ is an eigenvalue of A:*

$$(\gamma + s) \in \sigma(A) \Leftrightarrow (\gamma - s) \in \sigma(A)$$

Proof 4 *From definition (1) $A^T = 2\gamma I_{2n} - JAJ^{-1}$ then*

$$\begin{aligned} P_A(s+\gamma) &= \det\left[(s+\gamma) I_{2n} - A^T\right] = \det\left[(s+\gamma) I_{2n} - \left(2\gamma I_{2n} - JAJ^{-1}\right)\right] = \det\left[(s-\gamma) I_{2n} + JAJ^{-1}\right] \\ &= (-1)^{2n} \det[J] \det[-(s-\gamma) I_{2n} - A] \det\left[J^{-1}\right] = \det[(\gamma - s) I_{2n} - A] = P_A(\gamma - s) \blacksquare \end{aligned}$$

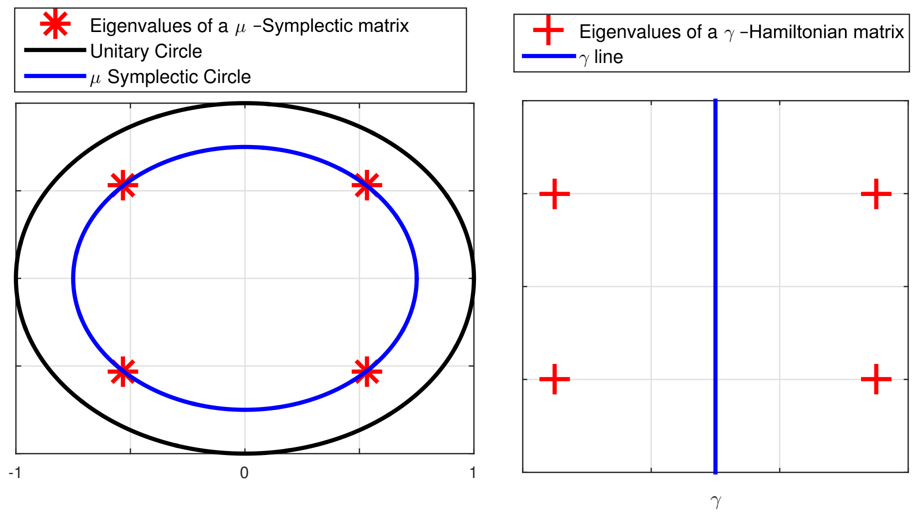


Figure 1: Eigenvalues of μ-Symplectic matrix and $\gamma-$ Hamiltonian matrix

The lemma states that the eigenvalues of A are symmetric with respect the vertical line $\mathbf{Re}(s) = \gamma$ in the complex plane, since the matrix A is real $\overline{s} + \gamma$ and $\overline{s} - \gamma$ are eigenvalues as well. Then the eigenvalues of γ-Hamiltonian matrix are placed: *i)* in quadruples symmetrically with respect the real axis and the line $\mathbf{Re}(s) = \gamma$, *ii)* purely imaginary pairs on the line $\mathbf{Re}(s) = \gamma$ and symmetric with the real axis *iii)* real pairs symmetric with the line $\mathbf{Re}(s) = \gamma$. See Fig. 1.

Remark 5 *The characteristic polynomial of $A \in \mathbb{R}^{2n\times 2n}$ $P_A(s)$ depends of n coefficients only.*

Proof 6 *The proof of the last remark is constructive, from definition (1) $A^T = 2\gamma I - JAJ^{-1}$ thus:*
$P_A(s) = \det\left(sI_{2n} - \left(2\gamma I - JAJ^{-1}\right)\right) = \det\left((s-2\gamma)I_{2n} + JAJ^{-1}\right) = \det(J)\det\left((s-2\gamma)I_{2n} + A\right)\det\left(J^{-1}\right)$
$= \det\left((s-2\gamma)I_{2n} + A\right) = P_L(2\gamma - s)$
then

$$P_A(s) = s^{2n} + a_{2n-1}s^{2n-1} + ...a_1 s + a_0 = (2\gamma - s)^{2n} + a_{2n-1}(2\gamma - s)^{2n-1} + ...a_1(2\gamma - s) + a_0 = P_A(2\gamma - s)$$

for $n = 1$

$$a_1 = -2\gamma \qquad a_0 = a_0$$

$n = 2$

$$a_3 = -4\gamma \qquad a_2 = a_2 \qquad a_1 = 8\gamma^3 - 2\gamma a_2 \qquad a_0 = a_0$$

$n = 3$

$$a_5 = -6\gamma \qquad a_4 = a_4 \qquad a_3 = 40\gamma^3 - 4\gamma a_4 \qquad a_2 = a_2 \qquad a_1 = -96\gamma^5 + 8\gamma^3 a_4 - 2\gamma a_2 \qquad a_0 = a_0$$

$n = 4$

$$\begin{aligned} &a_7 = -8\gamma && a_6 = a_6 && a_5 = 112\gamma^3 - 6\gamma a_6 && a_4 = a_4 \\ &a_3 = -896\gamma^5 + 40\gamma^3 a_6 - 4\gamma a_4 && a_2 = a_2 && a_1 = 2176\gamma^7 - 96\gamma^5 a_6 + 8\gamma^3 a_4 - 2\gamma a_2 && a_0 = a_0 \end{aligned}$$

$\vdots$ ∎

Then the coefficients of $2n$ degree characteristic polynomial $P_A(s)$ can be rewritten such that $P_A(s)$ depends of n coefficients only. Of course if $\gamma = 0$ the characteristic polynomial of Hamiltonian matrix is obtained. Making the substitution $\phi = s - \gamma$ reduces $P_A(s)$ to an even polynomial, for instance: if $n = 1 \rightarrow P_A(\phi) = \phi^2 - \gamma^2 + a_0$ and $n = 2 \rightarrow P_A(\phi) = \phi^4 + \left(a_2 - 6\gamma^2\right)\phi^2 + 5\gamma^4 - a_2\gamma^2 + a_0$

μ-Symplectic Matrices

Definition 7 $\tilde{M} \in \mathbb{R}^{2n\times 2n}$ *is called Symplectic if*

$$\tilde{M}^T J \tilde{M} = J$$

is satisfied.

The determinant of any Symplectic matrix is one, $\det\left(\tilde{M}\right) = 1$ [1], thus $\tilde{M}$ is not singular. If $A, B \in \mathbb{R}^{2n\times 2n}$ are Symplectic then AB is Symplectic too, $(AB)^T J (AB) = B^T A^T J (AB) = B^T J B = J$, and from the definition $\tilde{M}^{-1} = J^{-1}\tilde{M}^T J$ is also Symplectic $\left(J^{-1}\tilde{M}^T J\right)^T J \left(J^{-1}\tilde{M}^T J\right) = J^T \tilde{M} J \tilde{M}^T J = J^T \tilde{M}\left(\tilde{M}^{-1}\tilde{M}\right) J\tilde{M}^T J = J^T \underbrace{\tilde{M} J \tilde{M}^T}_{J} J = J^T J J = J$. Therefore the set of Symplectic matrices from a Group.

Lemma 8 *The characteristic polynomial of Symplectic matrix $P_{\tilde{M}}(\lambda)$ can be expressed as:*

$$P_{\tilde{M}}(\lambda) = \lambda^{2n} P_{\tilde{M}}\left(\frac{1}{\lambda}\right)$$

Proof 9

$$\begin{aligned} P_{\tilde{M}}(\lambda) &= \det\left[\lambda I_{2n} - \tilde{M}^T\right] = \det\left[\lambda I_{2n} - J\tilde{M}^{-1}J^{-1}\right] \\ &= (-1)^{2n}\det\left[M^{-1}\right]\det\left[\left(\frac{I_{2n}}{\lambda} - M\right)\lambda\right] = \lambda^{2n}\det\left[\frac{I_{2n}}{\lambda} - M\right] = \lambda^{2n} P_{\tilde{M}}(\lambda) \quad \blacksquare \end{aligned}$$

The last proof states that $P_{\tilde{M}}(\lambda) = \lambda^{2n} + \tilde{m}_{2n-1}\lambda^{2n-1} + \ldots + \tilde{m}_1\lambda + 1$ is a reciprocal polynomial, this is equivalent to say that the coefficients satisfies the relation $P_{\tilde{M}}(\lambda)$ are $\tilde{m}_k = \tilde{m}_{2n.-k}$. Since $\tilde{M}$ is real if λ is an eigenvalue then $\bar{\lambda}$, λ^{-1} and $\bar{\lambda}^{-1}$ are as well eigenvalues. Furthermore, the eigenvalues are symmetric with respect to the unit circle.
An important characteristic of Symplectic matrices is the symmetry, due to the symmetry the $2n$ degree characteristic polynomial $P_{\tilde{M}}(\lambda)$ is reduced to n degree auxiliary polynomial (reduced polynomial) [6]. Let be

$$\rho = \lambda + \frac{1}{\lambda} \tag{2}$$

the Howard transformation, thus

$$\lambda^2 - \rho\lambda + 1 = 0$$

if $n = 2$

$$P_{\tilde{M}}(\lambda) = \lambda^4 + \tilde{m}_3\lambda^3 + \tilde{m}_3\lambda^2 + a\lambda + 1 \Longrightarrow P_{\tilde{M}}(\rho) = \rho^2 + \tilde{m}_3\rho + \tilde{m}_2 - 2$$

The next definitions and lemmas are a generalization of the Symplectic matrices [1, 6].

Definition 10 *$M \in \mathbb{R}^{2n\times 2n}$ is called μ-Symplectic matrix if the equation:*

$$M^T J M = \mu J$$

is satisfied for $\mu \in (0, 1]$

Lemma 11 *The determinant of a $M \in \mathbb{R}^{2n\times 2n}$ μ-Symplectic matrix is μ^n.*

Proof 12 $\det\left[M^T J M\right] = \det\left[M^T\right]\det\left[J\right]\det\left[M\right] = \det\left[\mu J\right] = \mu^{2n} \to \left(\det\left[M\right]\right)^2 = \mu^{2n}$ $\blacksquare$

Lemma 13 *The characteristic polynomial of μ-Symplectic matrix $P_M(\lambda)$ can be expressed as:*

$$P_M\left(\frac{\mu}{\lambda}\right) = \frac{\mu^n}{\lambda^{2n}} P(\lambda) \tag{3}$$

Proof 14

$$\begin{aligned} P_M(\lambda) &= \det\left[\lambda I_{2n} - M^T\right] = \det\left[\lambda I_{2n} - \mu J M^{-1} J^{-1}\right] = \det\left[\frac{\lambda}{\mu} M - I_{2n}\right]\det\left(\mu M^{-1}\right) \\ &= \mu^n \det\left[\left(-\frac{\lambda}{\mu}\right)\left(\frac{\mu}{\lambda} I_{2n} - M\right)\right] = \mu^n\left(-\frac{\lambda}{\mu}\right)^{2n}\det\left[\frac{\mu}{\lambda} I_{2n} - M\right] = \frac{\lambda^{2n}}{\mu^n} P_M\left(\frac{\mu}{\lambda}\right) \blacksquare \end{aligned}$$

Remark 15 *If $M \in \mathbb{R}^{2n\times 2n}$ is μ-Symplectic , then $\lambda \in \sigma(M) \Rightarrow \left(\frac{\mu}{\lambda}\right) \in \sigma(M)$. There are n pairs of eigenvalues and $\det\left[M\right] = \mu^n$ then if all eigenvalues have the same magnitude $\mu^n = \prod_{i=1}^{2n}|\lambda| = \prod_{i=1}^{2n}\left|re^{\theta_i}\right| = r^{2n} \to r = \sqrt{\mu}$ thus we can say that the eigenvalues of M are symmetric with respect to the circle of radius $r = \sqrt{\mu}$ and this fact is independent of n. See Fig. 1.*

Let be $P_M(\lambda) = m_{2n}\lambda^{2n} + ...m_1\lambda + m_0$ the characteristic polynomial of M, then from (3)

$$\begin{aligned} \lambda^{2n}\mu^{-n}P_M\left(\frac{\mu}{\lambda}\right) &= P(\lambda) \\ \lambda^{2n}\mu^{-n}\left[m_{2n}\left(\frac{\mu}{\lambda}\right)^{2n} + ...m_1\left(\frac{\mu}{\lambda}\right) + a_0\right] &= m_{2n}\lambda^{2n} + ...m_1\lambda + m_0 \end{aligned}$$

the following relations are satisfied

$$\begin{aligned} &\mu^n = m_0 \\ &\mu^{n-1}m_{2n-1} = m_1 \\ &\mu^{n-2}m_{2n-2} = m_2 \\ &\quad\vdots \\ &\mu^{-(n-2)}m_2 = m_{2n-2} \\ &\mu^{-(n-1)}m_1 = m_{2n-1} \\ &\mu^{-n}m_0 = m_{2n} \end{aligned} \tag{4}$$

hence the characteristic polynomial $P_M(\lambda)$ depends of n coefficients only, for instance if $n = 2$

$$P_M(\lambda) = \lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_3\mu\lambda + \mu^2 \tag{5}$$

Remark 16 *Applying transformation*

$$\boxed{\rho = \lambda + \tfrac{\mu}{\lambda}} \tag{6}$$

the $2n$ degree polynomial $P_M(\lambda)$ is transformed to a reduced polynomial $P_M(\rho)$ of n degree.

For instance if $n = 2$

$$P_M(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + a\mu\lambda + \mu^2 \tag{7}$$

$$P_M(\rho) = \rho^2 + a\rho + b - 2\mu \tag{8}$$

γ-Hamiltonian Systems

Any Linear Hamiltonian System can be written as:

$$\dot{y} = JH(t)\,y$$

with $H^T(t) = H(t) \in \mathbb{R}^{2n\times 2n}$, one of the most important properties of any linear Hamiltonian system is that its state transition matrix is Symplectic [2]. On the other hand, if A is a γ-Hamiltonian matrix then $A + \gamma I$ is Hamiltonian matrix, if $\dot{x} = [A(t) + \gamma I] = JH(t)\,x$ for $H(t) = H(t)^T$ thus $A(t) = J[H(t) + \gamma J]$

Definition 17 *Let be*

$$\dot{x} = A(t)\,x = J[H(t) + \gamma J]\,x \tag{9}$$

for some $H(t) = H(t)^T \in \mathbb{R}^{2n\times 2n}$ where $A(t) \in \mathbb{R}^{2n\times 2n}$ γ-Hamiltonian matrix. Then the system (9) is named γ-Hamiltonian system.

Theorem 18 *The state transition matrix of γ-Hamiltonian system (9) is μ-Symplectic with $\mu = e^{2\gamma t}$*

Proof 19 *Let be $N(t) = \Phi(t, 0)$ the state transition of (9) then*

$$\frac{d}{dt}N(t) = A(t)\,N(t)$$

on the other hand (for simplicity sake of notation we omit the dependence of time in the matrices A and N)

$$\frac{d}{dt}N^TJN = \dot{N}^TJN + N^TJ\dot{N} = (AN)^T JN + N^TJ(AN) = N^T\left(A^TJ + JA\right)N = 2\gamma N^TJN$$

$$\frac{d}{dt}N^TJN = 2\gamma N^TJN \tag{10}$$

since [1] $N^T(0)\,JN(0) = J$ *and from (10) we obtain*

$$N^T(t)\,JN(t) = e^{2\gamma t}J = \mu J$$

therefore N is μ-Symplectic. ■

Corollary 20 *If the γ-Hamiltonian system is τ-periodic, i.e.*

$$\dot{x} = J[H(t) + \gamma J]\,x \qquad H(t) = H(t + \tau) \tag{11}$$

with $H^T(t) = H(t)$, then the monodromy matrix of (11) M is μ-Symplectic

[1] The the matrix product $\left(\frac{d}{dt}N^TJN\right)N^TJN = N^TJN\left(\frac{d}{dt}N^TJN\right)$ is commutative $\left(2\gamma N^TJN\right)N^TJN = N^TJN\left(2\gamma N^TJN\right)$ $2\gamma N^TJNN^TJN = 2\gamma N^TJNN^TJN$

Damped Hill's Equation

Let be the vectorial Hill's Equation

$$\ddot{y} + D\dot{y} + (\alpha K + \beta B p(t))\, y = 0 \qquad p(t) = p(t+\tau) \tag{12}$$

with $y \in \mathbb{R}^n, \alpha, \beta \in \mathbb{R}$, $D = D^T > 0$, $K, B \in \mathbb{R}^{n\times n}$ constant matrices and $p(t)$ with zero average:

$$\frac{1}{\tau}\int_0^\tau p(t) = 0$$

we may express (12) as:

$$\dot{x} = \begin{bmatrix} 0 & I_n \\ -(\alpha K + \beta B p(t)) & -D \end{bmatrix} x \tag{13}$$

where $x = \begin{bmatrix} y & \dot{y} \end{bmatrix}^T$ if $A(t) = \begin{bmatrix} 0 & I_n \\ -(\alpha K + \beta B p(t)) & -D \end{bmatrix}$ then

$$\dot{x} = A(t)\, x \quad A(t) = A(t+\tau) \tag{14}$$

Remark 21 *There exist an transformation such that $D = S\tilde{D}S^T$ is diagonal [7], thus without loss of generality we assume $D = Diag\{d_1, d_2, ...d_n\}$, therefore the matrix $A(t)$ satisfied the definition of γ-Hamiltonian matrix with $K = K^T$, $B = B^T$ and $\gamma = \frac{1}{n}\sum_{i=1}^{n} d_i$, then the system (13) is γ-Hamiltonian*

0

Theorem 22 *[The Floquet Factorization] Considering the system (14), then the state transition matrix can be factorized as:*

$$\Phi(t, t_0) = P^{-1}(t)e^{R(t-t_0)}P(t_0)$$

where $P^{-1}(t) = \Phi(t, 0)e^{-Rt}$ in addition $P^{-1}(t) = P^{-1}(t+T)$. If $t_0 = 0$ then

$$\Phi(t, 0) = P^{-1}(t)e^{Rt} \tag{15}$$

The Floquet factorization defines a Lyapunov transformation [8, 9], as follows:

$$z(t) = P(t)x(t) \tag{16}$$

where $P(t) = e^{Rt}\Phi(0, t)$ then $\dot{z} = \dot{P}x + P\dot{x} = \left[\dot{P}P^{-1} + PAP^{-1}\right] z$
$= \left[\left(Re^{Rt}\Phi(0,t) + e^{Rt}\frac{d}{dt}\Phi(0,t)\right)P^{-1} + PAP^{-1}\right] z$
$= \left[\left(Re^{Rt}\Phi(0,t) - e^{Rt}\Phi(0,t)A\right)P^{-1} + PAP^{-1}\right] z = \left\{Re^{Rt}\Phi(0,t)\Phi(t,0)e^{-Rt}\right\} z = Rz$

$$\dot{z}(t) = Rz(t) \tag{17}$$

therefore any periodic system (14) always can transform into a time invariant system[2](17). Since (16) is a Lyapunov transformation preserves the stability properties [8]. However, in order to make the transformation (16) it is necessary to know the analytic solution of (14) which generally is not possible, only in few cases as the Meissner equation [5, 11], or when the periodic function is a piece-wise linear function [12] or the Lamé's equation [13], but we always can use numerical methods to solve the system [14].

Theorem 23 *[Lyapunov-Floquet] Let be*

$$M = e^{R\tau} = \Phi(\tau, 0) \tag{18}$$

be the monodromy matrix [3] associated to (14) and their eigenvalues $\sigma(M) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ called the characteristic multipliers then the system (14) is:
i) Asymptotically stable $\Leftrightarrow \sigma(M) \subset \mathring{D} = \{z \in \mathbb{C} : |z| < 1\}$
ii) Stable $\Leftrightarrow \sigma(M) \subset \bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, and if $|\lambda_i| = 1$ it is a simple root of the minimal polynomial of M
iii) Unstable $\Leftrightarrow \exists\, \lambda_i \in \sigma(M) : |\lambda_i| > 1$ or $\sigma(M) \subset \bar{D}$ & $\exists\, |\lambda_i| = 1$ which is a multiple root of the minimal polynomial of M

[2]The matrix R is not always real [10]. In the sequel, we only use the spectrum of $\sigma(R)$

[3]The monodromy matrix M is the transition state matrix evaluated in one period and their eigenvalues do not depend on the time t, i.e. $\sigma(\Phi(T + t_1, t_1)) = \sigma(\Phi(T + t_2, t_2))$

Corollary 24 *Let be the eigenvalues of* R, $\sigma(R) = \{\rho_1, \rho_{2.......}\rho_n\}$ *called characteristic exponents, then the system (14) is:*
i) Asymptotically stable $\Leftrightarrow \sigma(R) \subset \mathring{Z} = \{z \in \mathbb{C} : \mathrm{Re}\,(z) < 0\}$
ii) Stable $\Leftrightarrow \sigma(R) \subset \bar{Z} = \{z \in \mathbb{C} : \mathrm{Re}\,(z) \leq 0\}$*, and if* $\mathrm{Re}\,(z_i) = 0$ *are simple roots of the minimal polynomial of* R
iii) Unstable $\Leftrightarrow \exists\ \mu_i \in \sigma(R) : |\mu_i| > 0$ *or* $\sigma(R) \subset \bar{Z}\ \&\ \exists\,\mathrm{Re}\,(z_i) = 0$ *which is a multiple root of minimal polynomial of* R

Proof 25 *See [8]* ■.

Remark 26 $M = e^{R\tau}$ *is a state transition matrix evaluated in one period and (14) is* γ*-Hamiltonian system, then by the theorem (18)* M *is* μ*-Symplectic . Furthermore, the matrix* R *is* γ*-Hamiltonian thus (17) is a time invariant* γ*-Hamiltonian system.*

Proof 27 $M = e^{R\tau}$ μ*-Symplectic* $\left(e^{R\tau}\right)^T J \left(e^{R\tau}\right) = \mu J$

$$\begin{aligned} e^{R^T\tau} &= \mu J e^{-R\tau} J^{-1} \\ &= \mu J \left\{ I_{2n} - R\tau + \frac{RR\tau^2}{2} - \frac{RRR\tau^3}{3!} + \frac{RRRR\tau^4}{4!} + ... + \frac{R^k\tau^4}{k!} + ... \right\} J^{-1} \\ &= \mu e^{-JRJ^{-1}\tau} = e^{2\gamma\tau} e^{-JRJ^{-1}\tau} \end{aligned}$$

then

$$\begin{aligned} R^T\tau &= 2\gamma\tau I_{2n} - JRJ^{-1}\tau \\ R^T J + JR &= 2\gamma J \end{aligned}$$

■

Using the properties of μ-Symplectic matrices developed in section 3 we can apply them to M. The system (14) is stable $\Leftrightarrow |\lambda| < 1$ but

$$\rho = \lambda + \frac{\mu}{\lambda} \tag{19}$$

and

$$\lambda^2 - \lambda\rho + \mu = 0$$

$$\lambda = \frac{\rho \pm \sqrt{\rho^2 - 4\mu}}{2} \tag{20}$$

then

$$|\lambda| \leq 1 \Leftrightarrow |\rho| \leq \sqrt{2\,(1+\mu)} \tag{21}$$

For damped Hill's Equation (13) with $n = 2$ we have

$$P_M\,(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + a\mu\lambda + \mu^2 \tag{22}$$

where

$$\begin{array}{l} a = -Trace\,(M) \\ b = \frac{1}{2}\left[a^2 - Trace\left(M^2\right)\right] \end{array} \tag{23}$$

and the reduced polynomial

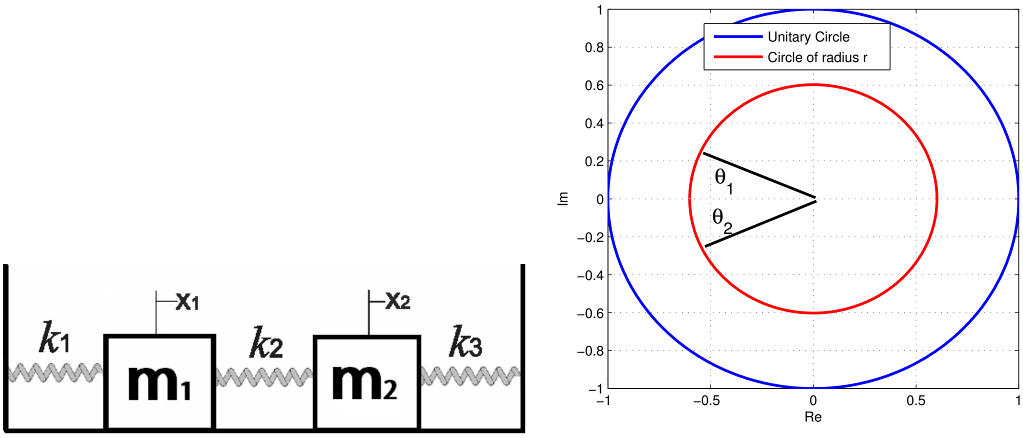
$$P_M(\rho) = \rho^2 + a\rho + b - 2\mu$$

$$\rho_{12} = \frac{-a \pm \sqrt{a^2 + 8\mu - 4b}}{2} \tag{24}$$

If (13) is stable $\rightarrow \lambda \in \mathbb{C},\ \lambda_1 = x - iy$, $\lambda_3 = x + iy$ and since the eigenvalues λ are symmetric with respect to the circle of radius $\sqrt{\mu}$

$$\begin{array}{l} \lambda_1 = x - iy \longrightarrow \lambda_2 = \frac{\mu}{\lambda_1} = \mu\,(x + iy) \\ \lambda_3 = x + iy \longrightarrow \lambda_4 = \frac{\mu}{\lambda_3} = \mu\,(x - iy) \end{array} \tag{25}$$

then is is easy to find de roots of (22). This is of great help, not only in the analysis of the damped Hill's equation, computationally it is not necessary to use numerical method to calculate the eigenvalues of the monodromy matrix, applying the relations (23) and the formulas (24) (20) it can compute the eigenvalues of M, this is due to symmetry of μ-Symplectic matrices.



(a) (b)

Figure 2: (a) Mechanical Model, (b) Characteristic multipliers position for attenuation

Theorem 28 *The Damped Hill's Equation (13) with $n = 2$ is Asymptotically stable if and only if the inequalities:*

$$\begin{aligned} b + (1+\mu)a + (1+\mu^2) &> 0 \\ b - (1+\mu)a + (1+\mu^2) &> 0 \\ \mu^2 + 4\mu + 1 - b &> 0 \\ \mu^4 + 2\mu^3 + 2\mu^2 + 2\mu + 1 + a^2\mu - b\,(\mu+1)^2 + &> 0 \end{aligned}$$

are satisfied.

Proof 29 *If*[4] $\lambda = \frac{s+1}{s-1}$ *then*

$$P(z) = \lambda^4 + a\lambda^3 + b\lambda^2 + a\mu\lambda + \mu^2$$

$P(s) = \left(\frac{s+1}{s-1}\right)^4 + a\left(\frac{s+1}{s-1}\right)^3 + b\left(\frac{s+1}{s-1}\right)^2 + a\mu\left(\frac{s+1}{s-1}\right) + \mu^2 = 0$
$P(s) = s^4\left(a\mu + 1 + b + \mu^2 + a\right) + s^3\left(-4\mu^2 + 2a + 4 - 2a\mu\right) + s^2\left(6\mu^2 - 2b + 6\right) + s\left(4 - 4\mu^2 - 2a + 2a\mu\right) + \left(b - a - a\mu + \mu^2 + 1\right)$ *therefore (13) is stable if* $\mathrm{Re}(s)$ *<0. Applying the Routh-Hurwitz Criterion we obtain the inequalities of the theorem.* ■

Attenuation of Vibrations

The mass-spring model of the Fig. 2 is modeled by

$$M\ddot{y} + D\dot{y} + \tilde{K}y = 0 \tag{26}$$

where $y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $M = M^T > 0$, $D = D^T > 0$, $\tilde{K} = \tilde{K}^T \in \mathbb{R}^{2\times2}$ are constant matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix};\ D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \tilde{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_3 + k_2 \end{bmatrix} \tag{27}$$

since $D > 0$ the system (26) always is asymptotically stable . If

$$k_i(t) = k_{i_a} + \beta k_{i_b} \cos(\omega t)$$

with $k_{i_a} > 0,\ |k_{i_a}| > |k_{i_b}|$ for $i = 1, 2, 3$ then (26) is transformed into time periodic system

$$\dot{x} = \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ -M^{-1}\left(K + \beta Bp\left(t\right)\right) & -M^{-1}D \end{bmatrix} x \tag{28}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$

$$K = \begin{bmatrix} k_{1_b} + k_{2_b} & -k_{2_b} \\ -k_{2_b} & k_{3_b} + k_{2_b} \end{bmatrix};\ B = \begin{bmatrix} k_{1_b} + k_{2_b} & -k_{2_b} \\ -k_{2_b} & k_{3_b} + k_{2_b} \end{bmatrix};\ p\left(t\right) = \cos\left(\omega t\right)$$

[4]The transformation is a mapping of the plane 'λ' into the plane of 's', all the points in the plane 'λ' that are inside of the unit circle will be mapped into the left half-plane of the plane 's', the points on the unit circle will be mapped in the imaginary axis and the points outside the unit circle on the right half-plane of the 's' plane.

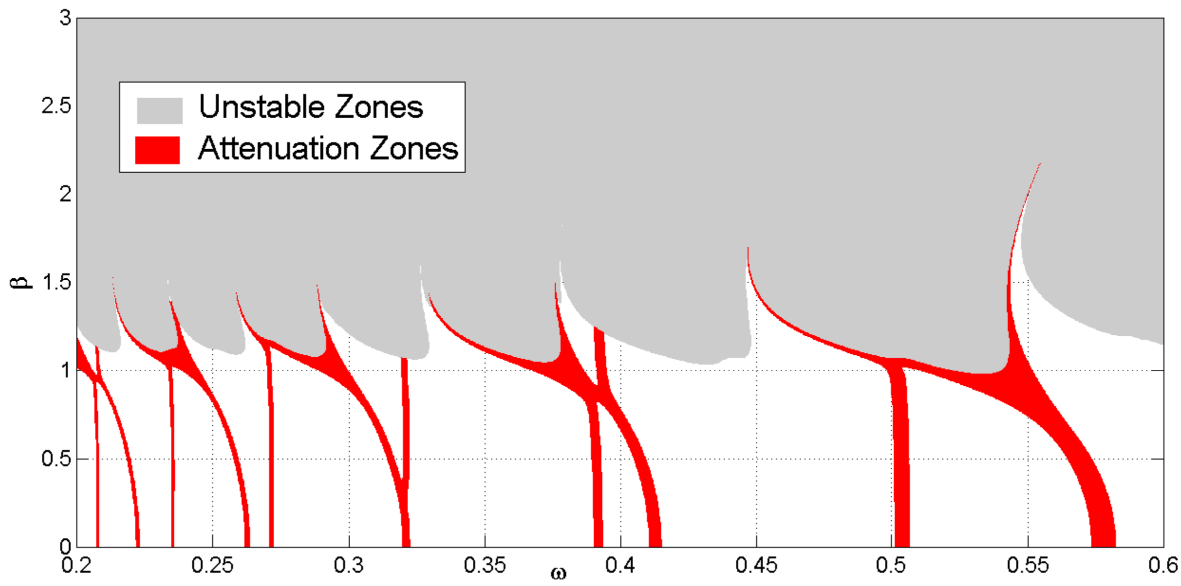


Figure 3: Stability Chart and attenuations zones of (28) (29)

Therefore the goal of this section is to show that for some parameters (ω, β) the responses of the system (28) are more attenuated than the responses of (26), to prove this assertion it is possible to compute Lyapunov exponents of (28) and (26) to compare the attenuation of the systems,see [11]. In previous works [3, 4] through the averaging and perturbations methods is concluded that the maximal attenuation occurs close to the called *anti-resonance frequencies* $\omega_k = \frac{\omega_2 - \omega_1}{k}$ where $\sigma(K) = \{\omega_1^2, \omega_1^2\}$.
It is known that for a second order system the real part of s is related to the attenuation factor. We have a fourth order system but due to the $\mu-$symmetry, it is equivalent to analyze the system as second order. If the system (28) is transformed into time-invariant γ-Hamiltonian system (17) then characteristic exponents ρ are symmetric with respect to the vertical line γ, then the better position to obtain attenuation occurs when the characteristic exponents are on the line γ and closer to the real axis, this is equivalent to say that the characteristic multipliers λ are on the circle of radius $r = \sqrt{\mu}$ and close to the negative part of the real axis. This condition correspond to the attenuations zones of a discrete system of second order [15]. The algorithm can be stated as finding the parameters (ω, β) such that characteristic multipliers λ are on the unit circle and one pair of them be close to the real axis, i.e its argument $\theta_1 < \theta < \theta_2$ see Fig. 2(b).
Of course this algorithm is a semi-analytical method because we need to compute by numerical methods the monodromy matrix, but after computation of the monodromy matrix, the numerical methods are no required, with the formulas (23) (24) (20) λ is computed. However, the described method does not allow finding a direct relation between the frequencies, which is a drawback. .

Example
For the matrices

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}; \ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \ K = \begin{bmatrix} 3.1 & -0.1 \\ -0.05 & 1.05 \end{bmatrix} \tag{29}$$

the parametric excitation is inserted in k_3 only, following the algorithm for $\theta_1 = \theta_2 = 8°$ we obtain the stability chart Fig. 3, where the gray zones are unstable while the red zones suggest where the attenuation occurs. Of curse, the red lines are thinner close to $\beta = 0$. Select (ω, β) inside of the red zones as example $\omega = 0.3095$, $\beta = 0.705$ and $\omega = 0.32175$, $\beta = 0.768$ and making a numerical simulation, the responses of the state are shown in Fig. 4. In the Fig. 5 can be observed the characteristic multipliers and exponents configuration position due to the algorithm.

Conclusions

The symmetry of the γ-Hamiltonian System was used to analyzed the damped Hill's equation. As a result of this analysis was feasible to know the better position of the characteristics multipliers and exponents to achieve an increase in dissipation by parametric excitation for the attenuation of vibrations in a mechanical system.

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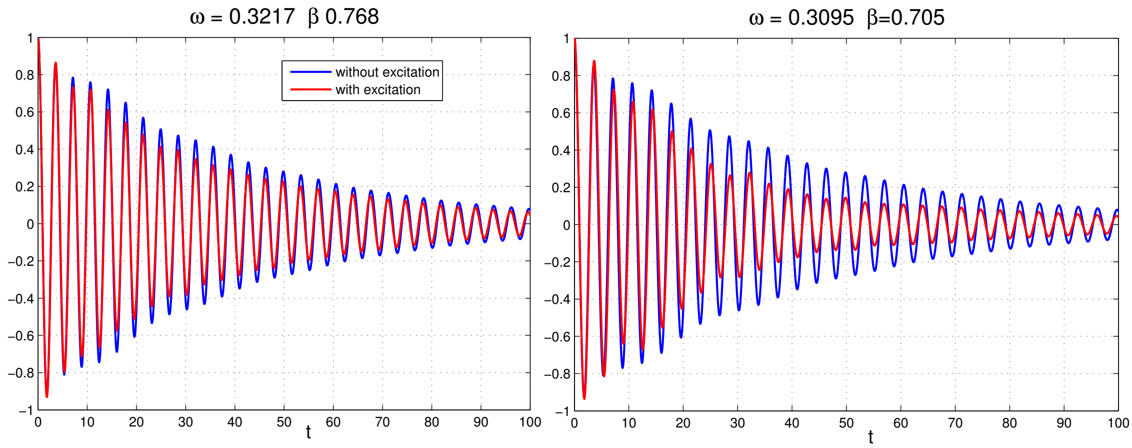

Figure 4: Examples of the response's signals of equations (28)(29) and (28) without parametric resonance

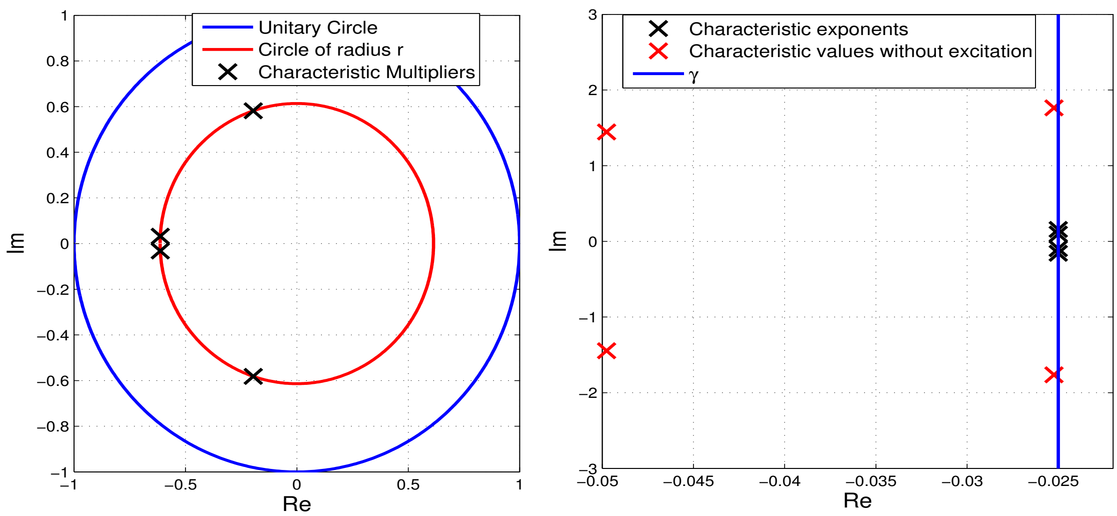

Figure 5: Characteristic multipliers and exponents for $\omega = 0.3217$ $\beta = 0.768$