# On a family of gradient-free control functions for extremum seeking problems

Victoria Grushkovskaya*,** and Christian Ebenbauer*
*Institute for Systems Theory and Automatic Control, University of Stuttgart, Germany
**Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine

*Summary*. In this paper, we construct a family of control functions which ensure the motion of a system in the direction of the negative gradient of a given cost function. It is assumed that the analytical expression of the cost function is unknown, and only its values are available for the control design. Such problem statement typically arises in extremum seeking control. Based on the Lie bracket approximation method, we show that the trajectories of the proposed system approximate the gradient flow of the cost function and prove the asymptotic stability of the extremum point. Several particular control strategies are compared. Moreover, we demonstrate the application of the obtained results to vibrational control.

## Introduction

The development of control strategies approximating the gradient of fully or partially unknown function is a challenging issue which finds applications in many problems of modern control theory. One of them is the extremum seeking problem which was studied in a number of papers (see, e.g., [1, 7]). In simple terms, a static extremum seeking problem can be formulated as follows: *suppose that the values of the cost function $J(x)$ are available, where $x \in \mathbb{R}^n$ is the state of a system and there exists $x^* \in \mathbb{R}^n$ which is a point of minimum of $J$: $J(x^*) < J(x)$ for all $x \neq x^*$. The purpose is to construct a control law that asymptotically steers a given system to $x^*$, assuming that only the values of the cost function $J(x(t))$ are available for control design.* Recently, it was proposed to use the Lie bracket approximation method to solve the extremum seeking problem [2]. Namely, it was shown in [2] that trajectories of a control-affine systems

$$\dot{x} = f_0(t,x) + \sum_{j=1}^{l} f_j(t,x)\sqrt{\omega}u_j(t,\omega t) \tag{1}$$

can be approximated by the trajectories of the following Lie bracket system:

$$\dot{\bar{x}}=f_0(t,\bar{x})+\frac{1}{T}\sum_{i<j}[f_i,f_j](t,\bar{x})\int_0^T\int_0^\theta u_j(t,\theta)u_i(t,\tau)d\tau d\theta. \tag{2}$$

In (1), (2), $x=(x_1,\dots,x_n)^T\in\mathbb{R}^n$, $\bar{x}=(\bar{x}_1,\dots,\bar{x}_n)^T\in\mathbb{R}^n$, $x(t_0)=\bar{x}(t_0)=x_0\in\mathbb{R}^n$, $\omega>0$, the vector fields $f_k:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, and $T$-periodic ($T>0$) control functions $u_j:[0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfy certain assumptions, and $[f_i,f_j](t,\bar{x})=\frac{\partial f_j(t,\bar{x})}{\partial x}f_i(t,\bar{x})-\frac{\partial f_i(t,\bar{x})}{\partial x}f_j(t,\bar{x})$. Moreover, it was proven that the asymptotic stability of a compact set $S\subset\mathbb{R}^n$ for system (2) implies the practical asymptotic stability of $S$ for system (1) (for the *practical stability* notions see, e.g., [2]). This result allows to approximate the gradient of a function without knowledge of it analytical expression. To illustrate this fact, consider a one-dimensional system $\dot{x}=J(x)\sqrt{\omega}\cos(\omega t)+\sqrt{\omega}\sin(\omega t),\ x\in\mathbb{R},\ \omega>0$. Considering $\cos(\omega t),\sin(\omega t)$ as controls, we see that the corresponding Lie bracket represents the gradient flow of $J$, namely, $\dot{\bar{x}}=\frac{1}{2}[J(\bar{x}),1]=-\frac{1}{2}J'(\bar{x})$ which converges to $x^*$ (locally or globally, based on the properties of $J(x)$).
Based on these results, several types of controls were proposed, see, e.g. [2, 6]. In this paper, we introduce a novel family of control functions which generalize the above strategies and allows to design new extremum seeking systems. Moreover, we show the application of the proposed functions to vibrational stabilization problems.

## Main results

Consider a system

$$\dot{x}=\sum_{i=1}^{n}\left(\Phi_i(J(x))\sqrt{\omega}_i\cos(\omega_i t)+\Psi_i(J(x))\sqrt{\omega}_i\sin(\omega_i t)\right), \tag{3}$$

where $x\in\mathbb{R}^n$, $\Phi_i,\Psi_i:\mathbb{R}\to\mathbb{R}^n$, $J:\mathbb{R}^n\to\mathbb{R}$, and $\omega_i=k_i\omega$ with some $\omega>0$, $k_i\in\mathbb{N}$, $k_i\neq k_j$ for $i\neq j$. The main result of this paper presents a family of vector fields $\Phi_i,\Psi_i$ such that the trajectories of (3) approximate the negative gradient flow of $J(x)$.

**Theorem 1.** *Let $J\in C^2(\mathbb{R}^n;\mathbb{R})$, and there exist $x^*\in\mathbb{R}^n : J(x)>J(x^*)$ and $\|\frac{\partial J(x)}{\partial x}\|^2>0$ for all $x\in B_\rho(x^*)\setminus\{x^*\}$, with $\rho>0$. Define $\Phi_i=\varphi_i e_i,\Psi_i=\psi_i e_i$, where $e_i$ denotes the $i$-th unit vector in $\mathbb{R}^n$, and $\varphi_i,\psi_i\in C^2(\mathbb{R};\mathbb{R})$ satisfy the relation*

$$\psi_i(z)=\varphi_i(z)\left(\int\frac{1}{\varphi_i(z)^2}dz+c\right), \tag{4}$$

*where $\int\frac{1}{\varphi_i(z)^2}dz$ is any antiderivative of the function $\frac{1}{\varphi(z)^2}$, and $c\in\mathbb{R}$ is an arbitrary constant. Then the point $x^*$ is locally practically uniformly asymptotically stable for system* (3). *Moreover, if the above conditions hold for every $\rho>0$ then $x^*$ is semi-globally practically uniformly asymptotically stable for system* (3).

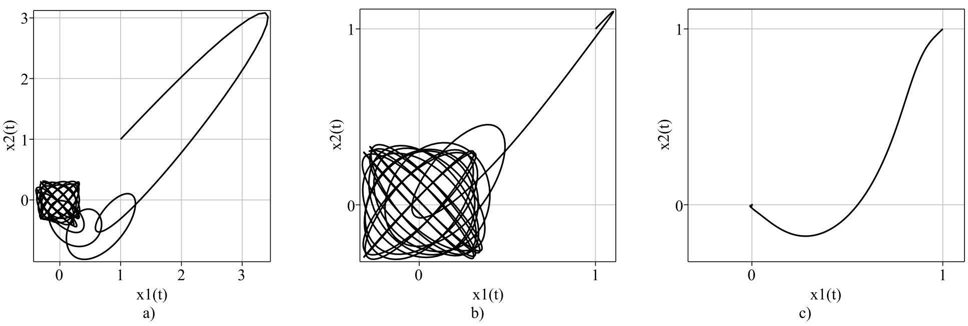


Figure 1: a)–c): the trajectories of system (3) for cases (5)– (7), respectively, $\omega_1 = 10$, $\omega_2 = 11$.

The proof of the Theorem 1 is based on the fact that, under conditions of the theorem, the Lie bracket system for (3) has the form $\dot{\bar{x}} = -\frac{1}{2}\frac{\partial J(\bar{x})}{\partial \bar{x}}$, and the asymptotic stability follows from the results of [2] (see also [3]).
The formula (4) generalizes some known results from the extremum seeking theory. For simplicity, assume $x \in \mathbb{R}$, and $\Phi_1 = \varphi$, $\Psi_1 = \psi$. The paper [2] presents the extremum seeking strategy of form (3) with

$$\varphi = J(x),\ \psi \equiv 1, \tag{5}$$

which has a simple form and can be used for a broad class of functions $J$. The control with so-called bounded updated rates were introduced in [6]. It is given by

$$\varphi = \sin(J(x)),\ \psi = \cos(J(x)). \tag{6}$$

The following functions also have bounded updated rates and possesses the useful property $\varphi(J(x)), \psi(J(x)) \to 0$ as $J(x) \to J(x^*)$, which ensure better behavior of solutions in cases of known extremum value $J(x^*)$ as it is, for example, the case in vibrational stabilization problems:

$$\varphi = \sqrt{\frac{1 - e^{J(x^*)-J}}{1 + e^{J-J(x^*)}}}\sin(e^{J-J(x^*)}+2\ln(e^{J-J(x^*)}-1)),\ \psi = \sqrt{\frac{1 - e^{J(x^*)-J}}{1 + e^{J-J(x^*)}}}\cos(e^{J-J(x^*)}+2\ln(e^{J-J(x^*)}-1)). \tag{7}$$

Note that the functions in (7) do not satisfy assumptions of Theorem 1 since they are not twice continuously differentiable at 0, however, the practical asymptotic stability in this case can be proven, see [3] for a complete proof and extensions. Moreover, practical *exponential* stability can be established under additional assumption on $J(x)$. In Fig. 1, we illustrate the above-type control strategies for $x \in \mathbb{R}^2$ and $J(x) = x_1^2 + x_2^2$ which has the global minimum at $x = (0, 0)^T$. As can be observed, (7) performs very well in comparison with (5) and (6). The results obtained can be also applied for the stabilization of the control-affine system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R} \tag{8}$$

with unknown $f, g$ and known control Lyapunov function. Namely, the following vibration control law was proposed in [5, 4]: $u = V(x)\sqrt{\omega}\cos(\omega t) + 2\gamma\sqrt{\omega}\sin(\omega t)$, where $\gamma$ is a positive constant, and $V(x)$ is a control Lyapunov function for (8). The asymptotic stability was proved based on the Lie bracket approximation for (8) with such control: $\dot{\bar{x}} = f(\bar{x}) - \gamma g(\bar{x})(L_g V(\bar{x}))^T$, where $L_g V = \frac{\partial V}{\partial x} g$. Similarly to the extremum seeking problem, we may generalize the proposed class of controls using $u(V(x), t) = \varphi(V(x))\sqrt{\omega}\cos(\omega t) + \psi(V(x))\sqrt{\omega}\sin(\omega t)$, with $\varphi, \psi$ satisfying (4).

## Conclusions

In this paper, we have presented a family of control functions which allow to approximate the gradient flow of the cost functions with fully or partially unknown analytical expression. We considered applications of this result in the extremum seeking and vibrational control theory. Several particular control strategies have been discussed.

## References

[1] Ariyur K.B., Krstić M. (2003). Real-Time Optimization by Extremum Seeking Control. Wiley-Interscience.

[2] Dürr H.B., Stanković M.S., Ebenbauer C., Johansson K. (2013). Lie bracket approximation of extremum seeking systems. *Automatica* **49**: 1538–1552.

[3] Grushkovskaya V., Zuyev A., Ebenbauer C. On a class of generating vector fields for the extremum seeking problem: Lie bracket approximation and stability properties, submitted (arXiv:1703.02348).

[4] Michalowsky S., Ebenbauer C. (2013). Swinging up the Stephenson-Kapitza pendulum. *Proc. 52nd IEEE Conference on Decision and Control*: 3981–3987.

[5] Scheinker A. and Krstić M. (2013). Minimum-Seeking for CLFs: Universal Semiglobally Stabilizing Feedback Under Unknown Control Directions *IEEE Trans. Autom. Control.* **58**: 1107–1122.

[6] Scheinker A. and Krstić M. (2014). Extremum seeking with bounded update rates. *Systems & Control Letters* **63**: 25–31.

[7] Tan Y., Moase W.H., Manzie C., Nešić D., Mareels I.M.Y. (2010). Extremum seeking from 1922 to 2010. *Proc. 29th Chinese Control Conference*: 14–26.