Bifurcation analysis of Nonlinear Normal Modes with the Harmonic Balance Method

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Summary. This work presents a frequency-domain method based on the Harmonic Balance Method (HBM) to perform bifurcation and stability analysis of Nonlinear Normal Modes (NNM). To do so, a frequency phase condition has been adapted from time domain in order to fix the non-uniqueness of the solution of the autonomous equation of motion. Then, a small damping coefficient has been introduced in the equation of motion to make invertible the matrices used during the pseudo-arc length continuation process. Finally, a shifted quadratic eigenvalue problem has been used to perform stability and bifurcation analysis. The resulting HBM-based algorithm permits the continuation of NNMs, the precise computation of bifurcation points as well as branch switching.

NNM computation as periodic solution

The theory of Nonlinear Normal Modes has been introduced by Rosenberg [1] by defining an undamped unforced NNM. The computed NNM was considered as synchronous periodic solutions. Since then, this definition has been extended in order to take modal interactions into account [2]. To do so, the synchronicity condition has been removed. In the literature, numerous numerical methods for the computation of NNMs can be found [2, 3]. In this work, the extended definition of NNMs is considered. NNMs are computed using HBM+AFT and continuation methods.

Equation of motion

A unforced undamped nonlinear dynamical system with 2 degrees of freedom is considered, as described in [2, 4]

$$\mathbf{M}\ddot{\mathbf{x}}(t) + d\mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f_{nl}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0} \tag{1}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{f_{nl}} = \begin{bmatrix} \frac{1}{2}x_1^3(t) \\ 0 \end{bmatrix}$$

with $\mathbf{x}(t)$ the displacement vector, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ the generalized mass, damping and stiffness matrices, $\mathbf{f_{nl}}$ the vector of nonlinear forces. The artificial damping parameter d is introduced to obtain an invertible square Jacobian during the pseudo-arc length continuation process. During the continuation, d is naturally equal to zero and undamped periodic solutions are obtained. By applying the HBM method as in [5], the equation of motion in the frequency domain is obtained

$$\mathbf{R}(\mathbf{X}, \omega, d) = \mathbf{Z}(\omega, d)\mathbf{X} + \mathbf{Fnl}(\mathbf{X}) = \mathbf{0} \tag{2}$$

where $\mathbf{X}$ and $\mathbf{Fnl}$ are the vectors of Fourier coefficients of the displacement and nonlinear forces respectively, $\otimes$ stands for the Kronecker tensor product, ∇ represents the time derivative operator, and

$$\mathbf{Z}(\omega, d) = \omega^2\nabla^2 \otimes \mathbf{M} + d\omega\nabla \otimes \mathbf{C} + \mathbf{I}_{2H+1} \otimes \mathbf{K} = \mathrm{diag}(\mathbf{K}, \mathbf{Z}_1, ..\mathbf{Z}_j, .., \mathbf{Z}_H) \tag{3}$$

$$\mathbf{Z}_j = \begin{bmatrix} \mathbf{K} - j^2\omega^2\mathbf{M} & d\omega\mathbf{C} \\ -d\omega\mathbf{C} & \mathbf{K} - j^2\omega^2\mathbf{M} \end{bmatrix}$$

$$\nabla = \mathrm{diag}(\mathbf{0}, \nabla_1, ..., \nabla_j, ..., \nabla_H) \quad \text{with} \quad \nabla_j = j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4}$$

Phase condition

Since Eq. (1) is autonomous, if $\mathbf{x}(t)$ is solution then $\mathbf{x}(t + \Delta t)$ is also solution for any Δt. For periodic solutions, this non-uniqueness of the solution can be interpreted as an undetermined phase. Thus, one phase condition $r(\mathbf{X}) = 0$ must be introduced in order to prescribe the phase of the solution and therefore to establish the uniqueness of the solution [6]. A common phase condition consists in imposing the Fourier coefficient of one DOF equal to zero. In this work, a robust phase condition transposed from time domain to frequency domain is used instead.
In summary, a NNM is computed using the following augmented system coupled with pseudo-arc length method.

$$\begin{pmatrix} \mathbf{R}(\mathbf{X}, \omega, d) = \mathbf{Z}(\omega, d)\mathbf{X} + \mathbf{Fnl}(\mathbf{X}) \\ r(\mathbf{X}) \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2H+1} \\ 0 \end{pmatrix} \tag{5}$$

Stability and Bifurcation analysis of NNMs

Contrarily to Linear Normal Modes, NNMs present complex dynamical behaviour. Stability can change along their branches and bifurcations points such as Branch Points or Limit Points can exist. The stability is evaluated by means of Floquet exponents Λ computed with the following quadratic eigenvalue problem [5]

$$\mathbf{Q} = (\mathbf{R_X} + \Lambda\boldsymbol{\Delta}_1 + \Lambda^2\boldsymbol{\Delta}_2)\boldsymbol{\phi} = 0 \tag{6}$$

where $\boldsymbol{\phi}$ are complex eigenvectors, $\mathbf{R_X}$ is the Jacobian of (2) and

$$\boldsymbol{\Delta}_1 = 2\omega\nabla \otimes \mathbf{M} + \mathbf{I}_{2H+1} \otimes d\mathbf{C} = diag\left(d\mathbf{C}, \begin{bmatrix} d\mathbf{C} & 2\omega\mathbf{M} \\ -2\omega\mathbf{M} & d\mathbf{C} \end{bmatrix}, \dots, \begin{bmatrix} d\mathbf{C} & 2H\omega\mathbf{M} \\ -2H\omega\mathbf{M} & d\mathbf{C} \end{bmatrix}\right) \tag{7}$$

$$\boldsymbol{\Delta}_2 = \mathbf{I}_{2H+1} \otimes \mathbf{M} \tag{8}$$

Since the quadratic eigenvalue problem comes directly from the autonomous equation of motion [5], the non-uniqueness of the solution makes the Jacobian singular. Consequently, a null eigenvalue Λ with multiplicity 2 appears as a solution of the quadratic eigenvalue problem. In order to perform a stability analysis, these null eigenvalues are shifted according to [7]. The resulting shifted quadratic eigenvalue problem is then used to define augmented systems characterising Branch Points and Limits Points of NNMs. This is a transposition to NNMs of the augmented systems presented [5] for the bifurcation analysis of forced responses. Finally, a branch-switching algorithm is used to obtain the tangent vectors of the bifurcated branches. As can be seen in Fig. 1, all the NNM branches with their stability and bifurcations points have been computed up to the 9:1 internal resonance.

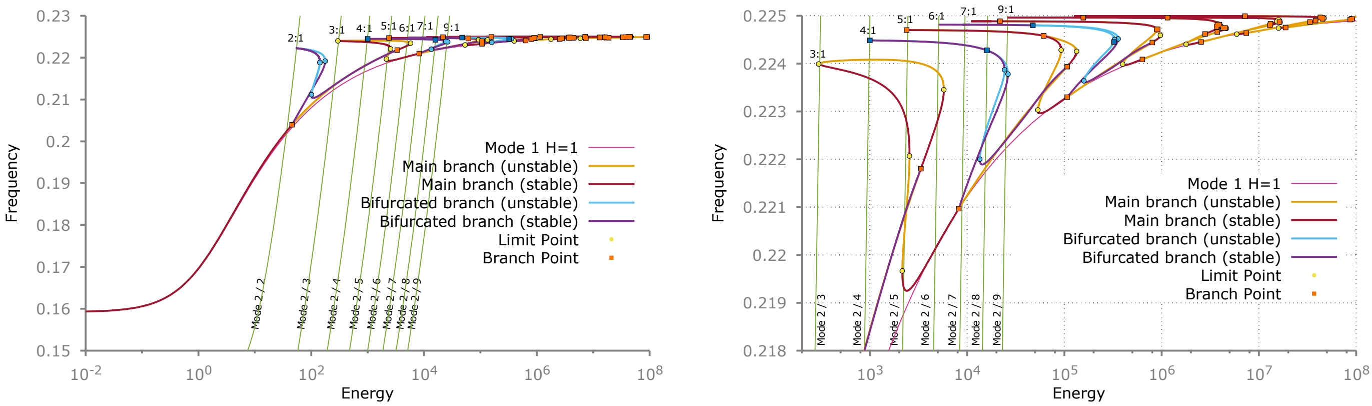

Figure 1: Left figure: NNM computed with H=20 harmonics. Right figure: Zoom on the internal resonances and bifurcated branches.

Conclusions

A bifurcation and stability analysis of NNMs has been developed using the HBM method. The introduction of a phase condition in frequency domain and a small damping coefficient leads to a more robust NNM computation. To compute the stability of the NNMs, a shifted quadratic eigenvalue problem is used. The bifurcations of NNMs are calculated using an augmented systems modified with respect to the shifted quadratic eigenvalue problem. The resulting bifurcation analysis method allows all the branches, their stability and their bifurcation points to be computed.

References

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