Beyond complete synchronization of identical systems : multidimentional dynamic consensus

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Summary. We propose an approach for synchronization analysis of nonlinear planar oscillators that allows, in particular, to treat a situation when the coupling gain is not strong enough to ensure (almost) global complete synchronization and coupled network behavior is caracterized by multistability.

During the last two decades the concept of synchronization became ubiquitous in various disciplines including engineering, physics, biology, computational neuroscience and many others. Generally speaking, synchronisation corresponds to the situation where some form of correlated behavior of a group or a network of systems appears as a result of local interconnections between the systems in the network. The general concept of synchronization of complex systems is extremely broad. Depending on the application domain and particular properties of the systems under consideration, different types of synchronization are considered - coordinate, phase and frequency synchronization, controlled synchronization to name a few. See, for example, Blekhman et al. (1997) for mathematical definitions of the synchronization phenomena.

Here we consider a synchronization problem for a network of nonlinear oscillators that are locally coupled using linear diffusive coupling. It is well known that in the case that the coupling is sufficiently strong and all the nodes are identical, the effect of complete synchronization appears in such a network, that is all the oscillators have the same phase, amplitude and frequency. Case of heterogeneous networks of oscillators was recently analyzed in Panteley et al. (2015) based on the two complementary paradigms: synchronization and collective (emergent) behavior. It was shown that if the coupling gain is strong, the oscillators *practically* synchronize (both in phase and amplitude). Moreover, the emergent collective behavior of the network is characterized by the dynamics of a single "averaged" oscillator. The notion of "dynamic consensus" was introduced in Panteley et al. (2015) to emphasize the dychotomy of these two effects.

Pursuing the same line of research, in this article we contemplate consequences of not strong coupling, that is, we consider linearly coupled networks of *identical oscillators* in the case when complete synchronization is not necessarily possible. In particular we consider a network of N Andronov-Hopf oscillators that are descibed by the following equations

$$\begin{aligned} \dot{z}_i &= f(z_i,\mu_i) + u_i, \quad i \in \mathcal{I} := \{1,\ldots,N\} \\ f(z_i,\mu_i) &:= -|z_i|^2 z_i + \mu_i z_i \end{aligned} \tag{1}$$

where z_i, $u_i \in \mathbb{C}$ are, respectively, the state and the input of ith oscillator, $\mu_i = \mu_{\text{R}i} + \mathrm{i}\mu_{\text{I}i} \in \mathbb{C}$ is a complex parameter which defines the asymptotic behavior of the ith oscillator. We assume that the graph of interconnections between the oscillators is connected and undirected and the coupling is linear, *i.e.*, the input u_i is given by

$$u_i = -\gamma\Big[d_{i1}(z_i - z_1) + d_{i2}(z_i - z_2)\ldots + d_{iN}(z_i - z_N)\Big], \quad i \in \mathcal{I} \tag{2}$$

where the constants d_{ij} $(i, j \in \mathcal{I})$ are positive and the scalar parameter $\gamma > 0$ corresponds to the coupling strength. Denoting by $\boldsymbol{z} \in \mathbb{C}^N$ the overall network's state, that is $\boldsymbol{z} = [z_1,\ldots,z_N]^\top$, using (1) and the expression for the diffusive coupling, (2), the overall network dynamics can be rewritten in the following form

$$\dot{\boldsymbol{z}} = F(\boldsymbol{z}) - \gamma L\boldsymbol{z}, \tag{3}$$

where matrix $L \in \mathbb{R}^{N\times N}$ is a so-called Laplacian matrix of the interconnections defined by the constants d_{IJ} and the function $F : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$F(\boldsymbol{z}) = [f(z_i,\mu_i)]_{i\in\mathcal{I}}\,, \tag{4}$$

Regrouping all the linear terms together we can rewrite the network model model as

$$\dot{\boldsymbol{z}} = A_\gamma \boldsymbol{z} - C(\boldsymbol{z})\boldsymbol{z}, \tag{5a}$$

$$A_\gamma := \mathcal{M} - \gamma L. \tag{5b}$$

where diagonal matrices $C(\boldsymbol{z})$ and $\mathcal{M}$ are defined as follows

$$C(\boldsymbol{z}) := \mathrm{diag}(|\boldsymbol{z}_1|^2,\ldots,|\boldsymbol{z}_N|^2), \quad \mathcal{M} := \mathrm{diag}(\mu_1,\ldots,\mu_N).$$

In the case that the network graph is connected and undirected the Laplacian matrix $L = L^\top \geq 0$ can be presented in the form

$$L = V\Lambda V^{-1}, \tag{6}$$

where $\Lambda \in \mathbb{C}^{N\times N}$ is a diagonal matrix whose elements correspond to the eigenvalues of L and columns of the matrices $V,\ V^* \in \mathbb{C}^{N\times N}$ correspond to the right and left eigenvectors of the Laplacian. Notice that without loss of generality the eigenvalues of matrix L can be ordered in decreasing order, that is, $\lambda_1(L) > \lambda_2(L) \geq \ldots \geq \lambda_N(L)$. Clearly, the matrix A_γ has the same eigenvectors as L, while the eigenvalues of the matrix A_γ are defined as

$$\lambda_i(A_\gamma) = \mu - \gamma\lambda_i(L), \quad i = 1, \ldots, N. \tag{7}$$

It is well know that if the coupling parameter γ satisfies the property $Re\lambda_2(A_\gamma) < 0$ then it is possible to show that the network of oscillators *completely* synchronize – see e.g. Pogromsky et al. (1999); Pham and Slotine (2007). We treat here the case where this inequality is not satisfied, that is $\lambda_2(A_\gamma) \geq 0$. It is worth repeating that in case where $\lambda_2(A_\gamma) \geq 0$ complete synchronization is not necessarily achievable and other effects such as clustering may appear (see Choe et al. (2010)). We assume that the matrix A_γ has m non-negative eigenvaluesuse and we use the matrix V to define a coordinate transformation.

Using the same approach as in Panteley et al. (2015), we start our analysis by decomposing the network dynamics in two parts: on one hand, the dynamics of the collective mean-field nodes network motion and, on the other, the dynamics of each individual unit of the network relative to the dynamics of the mean-field's. To do so we propose a change of coordinates that is similar to that Panteley et al. (2015) and is based on the decomposition of the matrix V in two parts $V = [V_1, V_2]$, where submatrix V_1 is composed from the eigenvectors corresponding to nonnegative eigenvalues of the matrix A_γ, while V_2 corresponds to negative eigenvalues.

Next, we project the network dynamics onto two orthogonal subspaces defined by the matrices V_1 and V_2 and introduce the new coordinates $z_m = V_1^* z$ which descibe dynamics of a "mean-field" network, while the coordinates $z_e = V_2^* z$ or, equivalently, $e = z - V_1 z_m$, correspond to the dynamics of each individual oscillator relative to the dynamics of the mean-field oscillators z_m.

As a result, we represent the network's dynamics as an interconnection of two systems - one of them (z_m) corresponds to the behavior of the reduced order network of mean-field oscillators while the second part corresponds to the dynamics of the synchronization errors.

The drastic difference between the case of complete synchronization and the case considered here is that in the first case dimension of mean-field dynamics, that is that of z_m coincides with the dimension of a single oscillator (see Panteley et al. (2015)) while now z_m represents several oscillators, therefore the mean-field dynamics corresponds to a dynamics of a reduced order network of oscillators.

We define the synchronisation errors manifold as

$$\mathcal{S} = \{\boldsymbol{e} \in \mathbb{C}^N : \boldsymbol{e}_1 = \boldsymbol{e}_2 = \ldots = \boldsymbol{e}_N = 0\} \tag{8}$$

and we analyze the stability properties of this manifold. We consider several cases when thus defined synchronization manifold is globally asymptotically stable under some addtional assumptions on the properties of the matrix V using a quadratic Lyapunov function $V(e) = \frac{1}{2}e^* e$. Moreover, in this case, the dynamics of the overall network is defined by the behavior of the reduced order network of mean-field oscillators which appears again to be a network of identical oscillators but with *nonlinear* coupling gains.

Our theoretical results are illustraded with numerical simulations that we do not present here due to space limitations. Future extentions of the apprach include other type of planar oscillators and extension to the systems where individual nodes have higher dimension.

References

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