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Reduction of Amplitude Fluctuations in Synchronized MEMS-Based Oscillators

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Summary. We investigate the dynamics of a closed-loop MEMS-based oscillator which is synchronized by a weak harmonic drive and demonstrate how one can exploit nonlinear behavior to reduce oscillator amplitude fluctuations at large operating amplitudes. This work was motivated by experimental observations that prompted an integrated theoretical and experimental study. Analytical predictions are based on an oscillator model that incorporates a resonator element with a cubic (i.e., Duffing type) nonlinearity, weak coupling to a clean external harmonic drive, and both thermal (white) and frequency (colored) noise terms that account for interactions of the resonator with its environment. The method of stochastic averaging is used to derive an expression for the rate of amplitude fluctuations induced by the noise sources, and the results predict a reduction in the amplitude fluctuations by two orders of magnitude due to the Duffing nonlinearity at large amplitudes of operation. The predictions are experimentally demonstrated using a closed-loop oscillator with a MEMS-based disk resonator coupled to a small external sinusoidal signal from a signal generator. The results show how one can avoid the well-known synchronization-induced amplitude fluctuations by operating at large amplitudes with a nonlinear resonator element in the oscillator.

Synchronization, as the roots of its name explain (*syn*, meaning the same or common, and *chronos*, meaning time), is the keeping of common time among oscillators. Perhaps the earliest and most familiar example is the synchronization of two pendulum clocks hung on the same wooden beam, which was described by Christiaan Huygens as "odd kind of sympathy". In this case, the clocks are weakly coupled through forces transmitted by the beam, resulting in the pendulums of the clocks swinging together in synchronization. A recent interest in synchronization has been raised by the desire to reduce frequency fluctuations in MEMS-base oscillators, for time-keeping technologies. Furthermore, in MEMS-based oscillators, frequency fluctuations are intensively investigated since they directly affect the stability of the oscillator and hence can degrade the oscillator performance. However, while synchronization dramatically improves frequency fluctuations, it also can increase amplitude fluctuations due to frequency-amplitude noise conversion that is induced by the synchronization [1]. We note that amplitude stability and sensitivity are also key factors in sensing applications and hence, due to synchronization, there is a trade-off between amplitude and frequency stability characteristics. Nevertheless, in what follows, we show that by considering a disk resonator with a Duffing nonlinearity and operating at large amplitude, the synchronization-induced amplitude fluctuations can be largely suppressed.

We consider a model for the system shown in Figure 1 which consists of a closed-loop self-sustained oscillator with a high-Q nonlinear (Duffing type) resonator element, thermal (additive) and frequency (multiplicative) noise sources, and an externally injected harmonic signal. In terms of the resonator displacement (x), the model yields

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x + \tilde{\alpha}x^3 = S(x,\dot{x}) + \tilde{E}\cos(\omega_E t) + \tilde{\xi}_1(t) + \tilde{\xi}_2(t)x, \tag{1}$$

where overdots denote derivatives with respect to time, S represents the saturated closed-loop feedback input (gains, phase shifts, etc.), $\tilde{\alpha}$ is the Duffing nonlinearity parameter, Q is the quality factor ($Q \gg 1$), ω_0 is the resonator natural frequency, $\tilde{E}$ and ω_E are the amplitude and frequency of the externally injected synchronizing signal, and $\tilde{\xi}_{1,2}$ are wide-sense stationary, zero-mean, additive and multiplicative noises. For simplicity we will consider the case of hard-limiting mechanism that sets the oscillator amplitude, where $S \approx s{\cdot}\mathrm{sgn}(\dot{x})$. The model is augmented by the following assumptions: (i) the noises and the external signal are relatively weak components of the signal $(||\tilde{E}||, ||\tilde{\xi}_1||, ||\tilde{\xi}_2 x|| \ll ||s||)$; (ii) the frequency of the external signal is close to the resonator natural frequency $(|\frac{\omega_E-\omega_0}{\omega_0}| \ll 1)$; (iii) the thermal (additive) noise source is broad-band; (iv) the frequency (multiplicative) noise source is narrow-band and its spectral density is centered around zero frequency, implying that we consider an oscillator that is lightly perturbed by the synchronizing

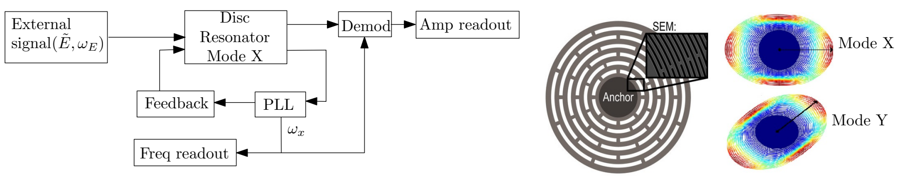

Figure 1: Setup for studying synchronization between the X mode of the disc resonator and an external signal, and schematic of the disc resonator along with the shapes of its two fundamental modes X and Y.

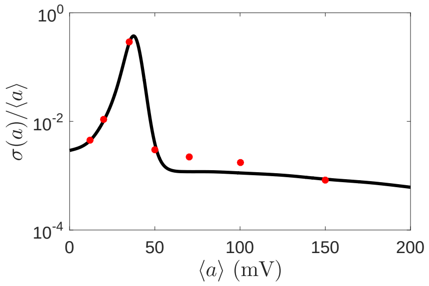

Figure 2: Normalized standard deviation, $\frac{\sigma(a)}{\langle a\rangle}$ ($\sigma(a) \equiv \sqrt{\langle a^2\rangle - \langle a\rangle^2}$), as a function of the mean (oscillator operating) amplitude, $\langle a\rangle$, as measured in millivolts. Comparison between model prediction (solid black line) and experimental measurements (red circles).

signal, thermal fluctuations with short correlation time, and frequency fluctuations with long correlation time. Using these assumptions, applying the method of stochastic averaging, normalizing the amplitude ($\tilde{a} = \frac{sQ}{\omega_0^2}a$) and time ($t = \frac{2Q}{\omega_0}\tau$), and linearizing the system around the stable synchronized locked-solution ($a = a^* + \delta a,\ \theta = \theta^* + \delta\theta$), we obtain the following pair of Langevin equations for the normalized deviations of the amplitude and phase

$$\dot{\delta a} = -\delta a - \frac{2\lambda_\theta(\Delta\tilde{\omega} + 3\alpha)}{1+9\alpha^2}\delta\theta + \left(S_{\xi_1}(\omega_0)\right)^{1/2}\eta_1, \tag{2}$$

$$\dot{\delta\theta} = \frac{3\alpha}{2}\delta a - \lambda_\theta\delta\theta - \xi_2 + \left(S_{\xi_1}(\omega_0)\right)^{1/2}\eta_2, \tag{3}$$

where $\lambda_\theta = \sqrt{\left(\frac{E}{2}\right)^2(1+9\alpha^2) - \Delta\tilde{\omega}^2}$, $\alpha = \frac{2s^2Q^3}{\omega_0^6}\tilde{\alpha}$, $E = \frac{\tilde{E}}{s}$, $\Delta\tilde{\omega} = \frac{2Q(\omega_0-\omega_E)}{\omega_0} + \frac{3\alpha}{2}$, $S_{\xi_1}(\omega) = 2\int_0^\infty\langle\xi_1(t)\xi_1(t+\tau)\rangle\cos(\omega\tau)d\tau$, $\langle\eta_{1,2}(t)\rangle = 0$, $\langle\eta_n(t)\eta_m(t+\tau)\rangle = \delta_{nm}\delta(\tau)$. As stated above, we assume the multiplicative noise, ξ_2, is narrow-band and hence its correlation time is large ($\tau_{c_2} \gg 1$). Furthermore, for a case of "strong" synchronization where $\lambda_\theta \gg 1$, the phase relaxation time, $\tau_\theta = \lambda_\theta^{-1}$, is small. Thus, if the magnitude of the multiplicative noise is much larger than its additive counterpart in Eq. (3), i.e., $||\xi_2|| \gg ||S_{\xi_1}(\omega_0)||^{1/2}$, we can set the time derivative on the left-hand side (LHS) of Eq. (3) to zero and obtain the following approximation $\delta\theta \approx \frac{1}{\lambda_\theta}\left(\frac{3\alpha}{2}\delta a - \xi_2\right)$. Substitution of this approximation into Eq. (2) yields a single equation for the amplitude fluctuations, $\dot{\delta a} = -\Gamma\delta a + \frac{2(\Delta\tilde{\omega}+3\alpha)}{1+9\alpha^2}\xi_2 + \left(S_{\xi_1}(\omega_0)\right)^{1/2}\eta_1$, where $\Gamma = 1 + \frac{3\alpha(\Delta\tilde{\omega}+3\alpha)}{1+9\alpha^2}$. Thus, the variance of the amplitude fluctuations can be readily calculated and given by $\langle\delta a^2\rangle = \frac{S_{\xi_1}(\omega_0)}{2\Gamma}\left[1-\exp(-2\Gamma\tau)\right] + \left(\frac{\Delta\tilde{\omega}+3\alpha}{1+9\alpha^2}\right)^2\frac{4D_{\xi_2}}{\Gamma(1+\tau_{c_2}\Gamma)}\left[1-\exp\left(\frac{-2\Gamma}{1+\tau_{c_2}\Gamma}\tau\right)\right]$, where the second term on the right-hand side (RHS) is obtained with the aid of a unified colored-noise approximation and D_{ξ_2} is the noise intensity of ξ_2. At steady-state, $t \to \infty$, the variance of amplitude fluctuations reduces to $\langle\delta a^2\rangle = \frac{S_{\xi_1}(\omega_0)}{2\Gamma} + \left(\frac{\Delta\tilde{\omega}+3\alpha}{1+9\alpha^2}\right)^2\frac{4D_{\xi_2}}{\Gamma(1+\tau_{c_2}\Gamma)}$. Hence, for a large Duffing nonlinearity, $\alpha \gg 1/3$, the steady-state variance reaches its minimal value due to two factors: (i) the suppression of additive noise as a result of nonlinear damping enhancement, described by the first term on the RHS, $\Gamma|_{\alpha\gg1/3}/\Gamma|_{\alpha=0} \to 2$, and (ii) the frequency-amplitude noise conversion is similarly reduced, as seen from the second term on the RHS, $\frac{\Delta\tilde{\omega}+3\alpha}{1+9\alpha^2}|_{\alpha\gg1/3} \to 0$.

The experimental device has natural frequency, $f_0 = \omega_0/2\pi = 251$(kHz), bandwidth of 2.8(Hz) which corresponds to a quality factor, $Q = 90,000$, and a dimensional Duffing nonlinearity, $\alpha_d = 1.85\times10^{-3}$(Hz/mV^2), where the oscillator frequency is measured as $f_{osc} = f_0 + \alpha_d|x|^2$, and $|x|$ is the oscillator amplitude measured in mV. The external signal is setup to have a relative magnitude that is 10% that of the oscillator amplitude and a frequency offset ,$\Delta f = f_0 - f_E$, ranging from 0.06(Hz) to 4.3(Hz) depending on the oscillator amplitude. The amplitude of the oscillator is varied from 12(mV) to 150(mV), which corresponds to a range of non-dimensional Duffing nonlinear coefficients, α. Note that the transition from linear to nonlinear behaviour, $\alpha \approx 1/3$, corresponds to an oscillator amplitude of 48.66(mV). Using the assumptions made in the model and comparing the measured spectrum of the amplitude fluctuations for the free-running and the synchronized oscillator, we obtain in a straightforward manner the noise levels of the additive and multiplicative noises, which yield $||\tilde{\xi}_1(\omega_0)||^2/(f_0^4\langle x\rangle^2) = 1.4\times10^{-7}$ and $||\tilde{\xi}_2||^2/f_0^4 = 1.54\times10^{-4}$. However, for the multiplicative noise, which assumed to be an Ornstein-Uhlenbeck process, the noise level is a combination of the noise intensity, $\tilde{D}_{\xi_2}$, and the correlation time, t_{c_2}. Thus, by setting $\tilde{D}_{\xi_2} = ||\tilde{\xi}_2||^2/f_0^4$, we find optimal agreement between the model and the measurements for a correlation time of $t_{c_2} = 1.215$(sec). Figure 2 show a comparsion between the model prediction and the experimental measurements. Note that in the nonlinear range (i.e., $\langle a\rangle > 48.66$(mV)), both the model and the measurements reveal a significant drop in the amplitude fluctuations, by two orders of magnitude. The most important feature of these results is that the Duffing nonlinearity enables a new and efficient way to clean amplitude fluctuations in synchronized oscillator, which together with the improved frequency fluctuations (due to synchronization), leads to a significant enhancement of the oscillator performance.

References

[1] Kurokawa, K. (1968). Noise in synchronized oscillators. IEEE Trans. Microw. Theory Techn., vol. 16, no. 4, pp. 234-240.