Sufficient conditions for convergence of discrete-time Lur'e type systems[1]

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Summary. Discrete-time Lur'e type systems are the interconnection between a discrete-time linear plant and a static nonlinearity satisfying sector conditions. The notion of convergence for such systems is investigated in this paper. We provide sufficient conditions based on two Lyapunov functions. These conditions are formulated as tractable matrix inequalities.

Introduction

The notion of convergence has been introduced in [1] for nonlinear continuous-time systems. A system is called convergent if two conditions are fulfilled: there exists a unique solution that is bounded on the whole set of times and this solution is globally asymptotically stable. This generic notion is of crucial interest for several issues in control system theory: output regulation, tracking, model reduction, frequency response functions for nonlinear systems. The literature is rich concerning sufficient conditions ensuring that a continuous-time nonlinear system is convergent [1, 6]. Nevertheless to the best authors' knowlegdge, only the paper [5] copes with the framework of discrete-time nonlinear systems. The aim of this paper is to provide sufficient conditions to ensure the convergence of discrete-time Lur'e systems.

Main result

Let us consider a general discrete-time nonlinear system defined by

$$x_{k+1} = f(x_k, w_k, k), \quad \forall k \in \mathbb{N}, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state and $w_k \in \mathbb{R}^m$ is the input. We assume that the function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{Z} \to \mathbb{R}^n$ is continuous with respect to the two first arguments for any third one. The notion of convergence is defined as follows.

Definition 1 *The discrete-time nonlinear system (1) is said to be (uniformly, exponentionally) convergent if $i)$ there exists, for any given bounded input w_k, a unique solution $\overline{x}_k^w$, called* steady-state solution*, that is defined and bounded on $\mathbb{Z}$ and $ii)$ $\overline{x}_k^w$ is globally (uniformly, exponentially) asymptotically stable.*

The system of the form (1) is too generic to allow tractable sufficient conditions to ensure its convergence. Here we will consider the particular class of discrete-time Lur'e type systems defined as the interconnection between a linear plant and a static nonlinearity and formalized by

$$x_{k+1} = Ax_k + B\varphi(y_k) + Fw_k, \quad y_k = Cx_k, \quad \forall k \in \mathbb{N}, \tag{2}$$

with $\varphi : \mathbb{R}^p \to \mathbb{R}^p$ a static decentralized nonlinearity [4] verifying the following cone bounded sector conditions concerning respectively the nonlinearity and its slope, for all positive definite diagonal matrices $R, S \in \mathbb{R}^{p\times p}$ and given positive definite diagonal matrices Ω and $\overline{\Omega}$:

$$\mathrm{SC}_{\Omega}(R, y) := \varphi(y)'R(\varphi(y) - \Omega y) \leq 0, \quad \forall y \in \mathbb{R}^p, \tag{3}$$

$$\overline{\mathrm{SC}}_{\overline{\Omega}}(S, y^a, y^b) := (\varphi(y^a) - \varphi(y^b))'S(\varphi(y^a) - \varphi(y^b) - \overline{\Omega}(y^a - y^b)) \leq 0, \quad \forall y^a, y^b \in \mathbb{R}^p. \tag{4}$$

Thanks to the condition (4), the nonlinearity φ is continuous and induces the continuity with respect to x_k and w_k of the map $f := Ax_k + B\varphi(Cx_k) + Fw_k$, which is independent on the time k. It should be also noticed that the nonlinearity φ is monotonic to avoid the existence of multiple fixed points of the system (2) preventing its convergence.
The convergence of a system is based on two main key points: incremental stability and the existence of a positively invariant compact set. The idea to reach these two crucial properties is to use two Lyapunov functions. The choice of the class of these Lyapunov functions should be suitable for discrete-time Lur'e systems and is inspired by the class of Lyapunov Lur'e functions introduced in [3, 2] and consisting of the sum of a quadratic term with respect to the state x_k and a cross term between the nonlinearity $\varphi(y_k)$ and the output y_k.
If there exist a symmetric positive definite matrix $P_1 \in \mathbb{R}^{n\times n}$, diagonal positive definite matrices $S_1, S_2, \Delta_1 \in \mathbb{R}^{p\times p}$ and a scalar $0 < \rho < 1$ such that

$$\begin{bmatrix} A' \\ B' \\ 0 \end{bmatrix} P_1 \begin{bmatrix} A' \\ B' \\ 0 \end{bmatrix}' - \begin{bmatrix} \rho P_1 & C'\overline{\Omega}(\rho\Delta_1 - S_1) & -A'C'\overline{\Omega}(\Delta_1 + S_2) \\ \star & 2S_1 & -B'C'\overline{\Omega}(\Delta_1 + S_2) \\ \star & \star & 2S_2 \end{bmatrix} < 0_{n+2p}, \tag{5}$$

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with $\star$ denoting a symmetric term, then by using the function $V_1(x^a, x^b) = (x^a - x^b)' P_1 (x^a - x^b) + 2(\varphi(Cx^a) - \varphi(Cx^b))' \Delta_1 \overline{\Omega} C(x^a - x^b)$, $\forall (x^a, x^b) \in \mathbb{R}^n \times \mathbb{R}^n$, we have the following properties. Firstly $V_1(x^a, x^b) > 0$ for $x^a \neq x^b$ thanks to the condition (4), see [2]. Secondly, pre- and post-multiplying Inequality (5) by a vector concatenating $(x_k^a - x_k^b)$, $(\varphi(y_k^a) - \varphi(y_k^b))$ and $(\varphi(y_{k+1}^a) - \varphi(y_{k+1}^b))$, we obtain $V_1(Ax_k^a + B\varphi(Cx_k^a), Ax_k^b + B\varphi(Cx_k^b)) - \rho V_1(x_k^a, x_k^b) - 2\overline{\mathrm{SC}}_{\overline{\Omega}}(S_1, y_k^a, y_k^b) - 2\overline{\mathrm{SC}}_{\overline{\Omega}}(S_2, y_{k+1}^a, y_{k+1}^b) < 0$, for $x_k^a \neq x_k^b$ that induces $V_1(x_{k+1}^a, x_{k+1}^b) - \rho V_1(x_k^a, x_k^b) < 0$ and finally the incremental stability of (2). In addition, if there exist a symmetric positive definite matrix $P_2 \in \mathbb{R}^{n \times n}$, diagonal positive definite matrices $S_3, S_4, \Delta_2 \in \mathbb{R}^{p \times p}$ and positive definite scalars τ_1, τ_2, c and δ such that

$$\begin{bmatrix} A' \\ B' \\ 0 \\ F' \end{bmatrix} P_2 \begin{bmatrix} A' \\ B' \\ 0 \\ F' \end{bmatrix}' - \begin{bmatrix} \tau_1 P_2 & C'\Omega(\tau_1 \Delta_2 - S_3) & -A'C'\Omega(\Delta_2 + S_4) & 0 \\ \star & 2S_3 & -B'C'\Omega(\Delta_2 + S_4) & 0 \\ \star & \star & 2S_4 & -(\Delta_2 + S_4)\Omega C F \\ \star & \star & \star & \tau_2 I_m \end{bmatrix} < 0_{n+2p+m} \quad (6)$$

$$c(-1 + \tau_1) + \tau_2 \delta \leq 0, \quad (7)$$

then by using the function $V_2(x) = x' P_2 x + 2\varphi(Cx)' \Delta_2 \Omega C x$, $\forall x \in \mathbb{R}^n$, it yields by pre- and post-multiplying Inequality (6) by a vector concatenating x_k, $\varphi(y_k)$, $\varphi(y_{k+1})$ and w_k and summing the result to (7), that it holds that $V_2(Ax_k + B\varphi(y_k) + Fw_k) - c - \tau_1(V_2(x_k) - c) - \tau_2(w_k' w_k - \delta) - 2\mathrm{SC}_\Omega(S_3, y_k) - 2\mathrm{SC}_\Omega(S_4, y_{k+1}) < 0$ for $x_k \neq 0$. Thanks to the S-procedure, we have the implication $V_2(x_{k+1}) \leq c$ if $V_2(x_k) \leq c$ and $w_k' w_k \leq \delta$ and hence the existence of a positively invariant compact set. The inequalities (5)–(7) are bilinear matrix inequalities, but linear when fixing the scalars ρ, τ_1, τ_2, c and δ.

Numerical illustration

Let us consider the numerical example with $n = 2$, $p = m = 1$, $\Omega = \overline{\Omega} = 1.1$, $w_k = \cos((k-1)\pi/3)$, $\forall k \in \mathbb{N}$,

$$A = \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & -0.8 \end{bmatrix}; \; B = \begin{bmatrix} -0.6 \\ 1.2 \end{bmatrix};$$

$$C = \begin{bmatrix} 2 & 0.27 \end{bmatrix}; \; F = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix};$$

and the nonlinearity φ defined as a deadzone by

$$\varphi(y) = \Omega \mathrm{sign}(y) \max(0, |y| - 1), \; \forall y \in \mathbb{R}.$$

For $\rho = 0.99$, $\tau_1 = 0.9$, $\tau_2 = 10$, $c = 200$ and $\delta = 1$, the inequalities (5)–(7) admit the following solution $\Delta_1 = 0.0752$; $\Delta_2 = 0.016$; $S_1 = 2.89$; $S_2 = 0.35$; $S_3 = 0.47$; $S_4 = 0.009$;

$$P_1 = \begin{bmatrix} 11.42 & 0.47 \\ 0.47 & 0.62 \end{bmatrix}; P_2 = \begin{bmatrix} 2.04 & 0.09 \\ 0.09 & 0.09 \end{bmatrix}.$$

The convergence of two distinct trajectories to the steady-state solution is emphasized on Figure 1.

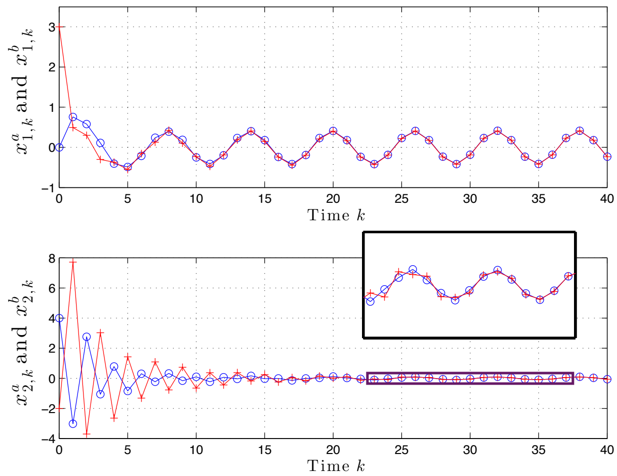

Figure 1: Trajectories starting from $x_0^a = (0\ 4)'$ and $x_0^b = (3\ -2)'$.

Conclusions

Sufficient conditions in terms of Lyapunov Lur'e functions has been provided for convergent discrete-time Lur'e type systems by ensuring incremental stability and the existence of a compact positively invariant set. Tractable LMIs have been presented for the existence of such Lyapunov functions.

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