Integral representation of fractional Euler-Lagrange equation with mixed boundary conditions

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Summary. In this paper we study the fractional Euler-Lagrange equation with mixed boundary conditions. This equation contain simultaneously the left Riemann-Liouville and the right Caputo derivatives. We transform the differential equation into the second kind Fredholm integral equation with adequate kernel. Finally, we present several plots of the kernel for different values of the order α.

Introduction

Nowadays, many scientists study differential and integral equations with fractional operators. They replace the classical (integer order) operators in equations by fractional operators. The resulting problems are solved using both analytical and numerical methods [4].
The second approach refers to a variational calculus, where the variational principle is modifies with replacing the integer order operators by a fractional one. Next, after the minimisation of action, we obtain the fractional Euler-Lagrange equations. These types of equations contain the fractional operator which is a composition of the left and right derivative. This fact causes problems in finding solutions of equations of a variational type and its poses a challenge for scientists [3]. In this paper we focus our attention on transformation the fractional Euler-Lagrange equation into an equivalent integral equation [1]. The analyzed equation is convert to the second kind Fredholm integral equation. The problem is studied with respect to mixed boundary conditions (Dirichlet and Neumann).

Fractional preliminaries

In this section, we introduce the fractional derivatives and integrals used in this work and some of their properties (see [2]). The left Riemann-Liouville derivative of order $\alpha \in (0,1)$ is defined as follows

$$D^{\alpha}_{a^+}\, x(t) = I^{1-\alpha}_{a^+} D\, x(t)\,, \tag{1}$$

and the right Caputo derivative of order $\alpha \in (0,1)$ has the form

$$^{C}D^{\alpha}_{b^-}\, x(t) = -I^{1-\alpha}_{b^-} D\, x(t)\,, \tag{2}$$

where D is an operator of the first order derivative and operators $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ are the left and right fractional integrals of order $\alpha > 0$, respectively, defined by

$$I^{\alpha}_{a^+} x(t) \;:=\; \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad (t > a)\,, \tag{3}$$

$$I^{\alpha}_{b^-} x(t) \;:=\; \frac{1}{\Gamma(\alpha)} \int_t^b \frac{x(\tau)}{(\tau-t)^{1-\alpha}} d\tau \quad (t < b)\,. \tag{4}$$

Main results

Let us consider the following fractional Euler-Lagrange equation

$$^{C}D^{\alpha}_{b^-} D^{\alpha}_{a^+} x(t) + \lambda x(t) = 0 \tag{5}$$

with mixed boundary conditions

$$x(a) = 0, \;\; D^{\alpha}_{a^+} x(t)\big|_{t=b} = 0 \tag{6}$$

By using the fractional integral operators $I^{\alpha}_{b^-}$ and $I^{\alpha}_{a^+}$ acting respectively on Eq. (5) we obtain

$$x(t) - x(a) - I^{\alpha}_{a^+}\left(D^{\alpha}_{a^+} x(t)\big|_{t=b}\right) - \lambda I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t) = 0 \tag{7}$$

Taking into account boundary conditions (6) we obtain

$$x(t) - \lambda I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t) = 0 \tag{8}$$

where

$$I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{1}{(t-s)^{1-\alpha}} \left(\frac{1}{\Gamma(\alpha)} \int\limits_s^b \frac{x(\xi)}{(\xi-s)^{1-\alpha}} d\xi \right) ds \tag{9}$$

The equation (8) can be written as the second kind Fredholm integral equation

$$x(t) = \lambda \int_a^b K^{(\alpha)}(t,s)\, x(s)\, ds \tag{10}$$

with kernel $K^{(\alpha)}(t,s)$ in the following form

$$K^{(\alpha)}(t,s) = \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \begin{cases} (t-a)^{2\alpha-1}\left(\frac{s-a}{t-a}\right)^{\alpha} \; {}_2F_1\left(1-\alpha, 1; 1+\alpha; \frac{s-a}{t-a}\right) & \text{if} s \le t \\ (s-a)^{2\alpha-1}\left(\frac{t-a}{s-a}\right)^{\alpha} \; {}_2F_1\left(1-\alpha, 1; 1+\alpha; \frac{t-a}{s-a}\right) & \text{if} s > t \end{cases} \tag{11}$$

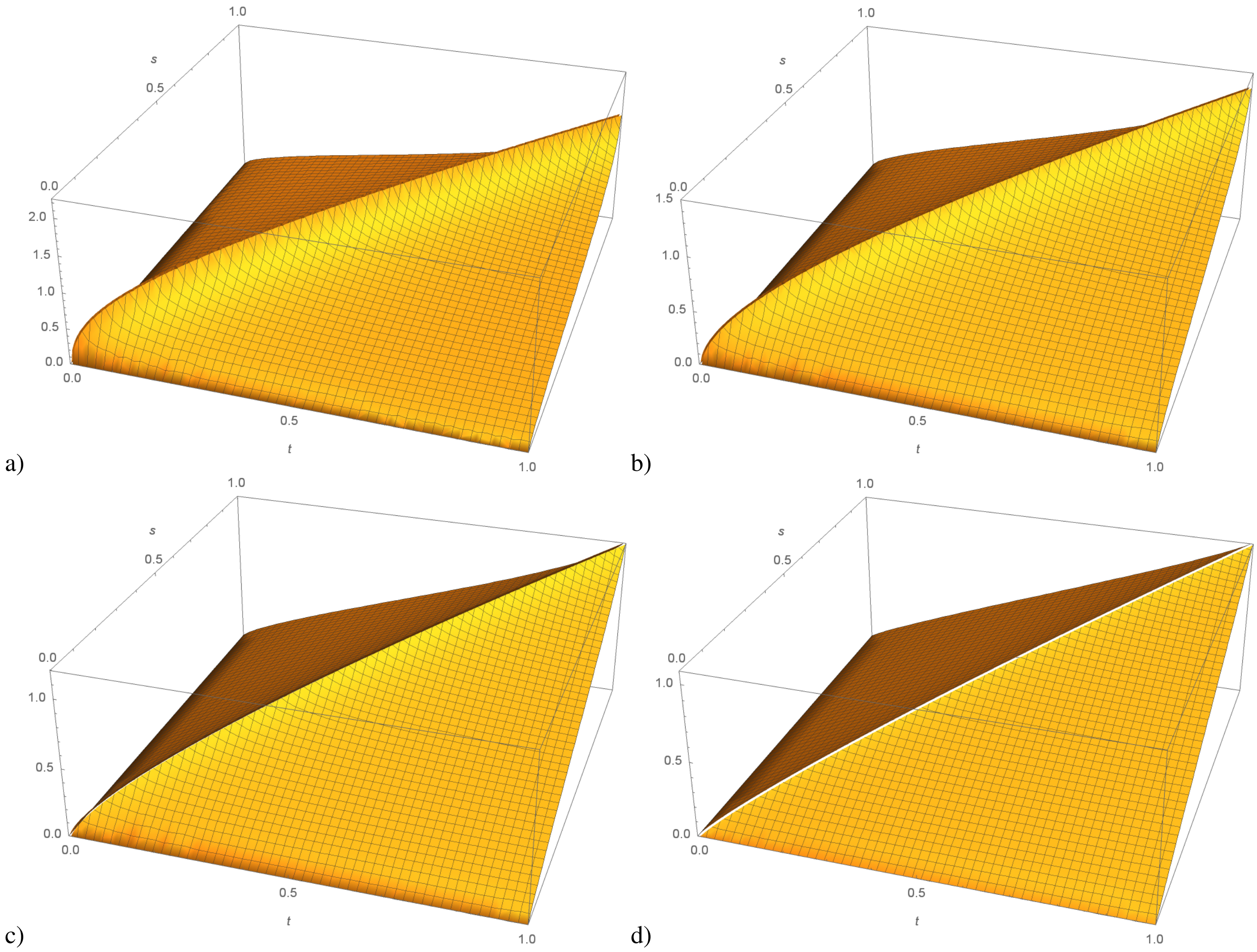

Figure 1: Plots of the kernel $K^{(\alpha)}(t,s)$ for: a) $\alpha = 0.6$, b) $\alpha = 0.7$, c) $\alpha = 0.8$, and d) $\alpha = 0.9$.

Conclusions

In this paper we studied the transformation of the fractional Euler-Lagrange equation into the second kind Fredholm integral equation with respect to mixed boundary conditions. Next, we present several plots of the kernel of the integral equation for different values of the order α. The presented results can be used in finding the analytical and numerical solutions of these types of equations.

References

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