Computing the viscous fluid flow between moving cylinders of an arbitrary cross-section

Petrov A.G.[*], Kazakova A.O.[**]
[*]*Moscow Institute of Physics and Technology (MIPT), Moscow, Russia*
[**] *Ulyanov Chuvash State University, Cheboksary, Russia*

Abstract. The algorithm of numerical research of a plane problem of the viscous fluid flow in the domain between two arbitrarily moving cylinders of an arbitrary cross-section is constructed. The Reynolds number is assumed to be small and the equations of the fluid motion are solved using the linear Stokes approximation. The biharmonic stream function is required. The Goursat representation and a modification of the boundary elements method are used to find it. The test examples confirm the high accuracy of the algorithm.

The studied problem has important applications in the theory of hydrodynamic lubrication in the theory of hydrodynamic stability and is also of interest for solving the problem of mixing.
The simplest solution can be obtained in the case of axisymmetric rotation of the concentric cylinders with constant angular velocities [1]. The exact solution for the case of the rotating eccentric cylinders can be obtained in the bipolar coordinates [2], [3]. In [4] the exact solution in the case of arbitrary motion of the eccentric cylinders is constructed.
This report proposes the algorithm of viscous fluid flow plane problem numerical investigation in the domain between two arbitrarily moving cylinders of an arbitrary cross-section.

Setting the problem and transition to the system of boundary equations

Hydrodynamic setting of the problem and the stream function

Let the fluid be contained between two cylindrical bodies of an arbitrary cross-section: in the plane $z = x + iy$ the contour ∂D_0 is located inside the contour ∂D_1. The components of velocity of the solid body point are developed of the components of the translational (v_x, v_y) and the angular velocity ω: $V_x = v_x - \omega\rho_y$, $V_y = v_y + \omega\rho_x$, where $\rho_x = x - x_\omega$, $\rho_y = y - y_\omega$, (x_ω, y_ω) are the coordinates of the point through which the axis of rotation passes.
If the fluid is incompressible, then the components of its velocity are expressed by the stream function which in the Stokes approximation satisfies the equation $\Delta^2\Psi = 0$, and it is possible to have a problem with the boundary conditions:

$$\left.\frac{\partial\Psi}{\partial n}\right|_{\partial D_k} = g_n^{(k)}(s) = (-1)^k\left(v_y^{(k)} + \omega_k\rho_x^{(k)}\right)\frac{dy_k}{ds} + (-1)^k\left(v_x^{(k)} - \omega_k\rho_y^{(k)}\right)\frac{dx_k}{ds},$$
$$\Psi\big|_{\partial D_k} = g^{(k)}(s) = (-1)^k\left(v_y^{(k)}x + \omega_k\frac{\rho_x^{(k)2}}{2} - v_x^{(k)}y + \omega_k\frac{\rho_y^{(k)2}}{2}\right) + C_k, \qquad k = 0,1. \tag{1}$$

where $x = x_k(s)$, $y = y_k(s)$ are the boundary equations ∂D_k, C_k are the undefined constants of integration.

The Goursat theorem and integral representations of harmonic functions

The Goursat theorem: any biharmonic in the domain D function Ψ can be expressed through two harmonic in the domain D functions φ and ψ: $\Psi = \varphi(x,y) + x\cdot\psi(x,y)$. The boundary conditions (1) take the form:

$$\varphi\big|_{\partial D_k} + x_k(s)\psi\big|_{\partial D_k} = g^{(k)}(s), \qquad \left(\partial\varphi/\partial n\right)\big|_{\partial D_k} + x_k(s)\left(\partial\psi/\partial n\right)\big|_{\partial D_k} + \psi\big|_{\partial D_k} dy_k/ds = g_n^{(k)}(s), \qquad k = 0,1. \tag{2}$$

Harmonic in D functions φ and ψ satisfy the following conditions, which can be used to find the constants C_0, C_1:

$$\int_{\partial D}\left(\partial\varphi/\partial n\right)ds = 0, \quad \int_{\partial D}\left(\partial\psi/\partial n\right)ds = 0. \tag{3}$$

Each of the harmonic functions φ and ψ satisfies the boundary integral equations of the form:

$$A\varphi_n(s) + B\varphi(s) = 2\pi\varphi(s) \quad \left(M \in \partial D_0\right), \quad A\varphi_n(s) + B\varphi(s) = 0 \quad \left(M \in \partial D_1\right), \tag{4}$$

where A and B are the linear operators:

$$A\varphi_n(s) = -\int_{\partial D} G(s,s')\varphi_n(s')ds', \quad B\varphi(s) = \int_{\partial D} G_n(s,s')\left(\varphi(s') - \varphi(s)\right)ds',$$

$G(s,s') = \ln\left(r(s,s')\right)$, $r^2(s,s') = (x-x')^2 + (y-y')^2$, $G_n(s,s') = \partial G/\partial n$.

The equations (2) and (3) are the system of eight equations with unknown functions $\varphi^{(k)}$, $\psi^{(k)}$, $\varphi_n^{(k)}$, $\psi_n^{(k)}$ ($k = 0,1$).

The numerical solution

For the numerical computation a modification of the boundary element method, the scheme without saturation described in details in [5], is used. The sampling of contours by the finite number of points $M_i^{(0)}$ ($i = \overline{1, N_0}$), $M_i^{(1)}$

($i = \overline{1, N_1}$) is introduced. Then the system of boundary equations (2), (4) reduces to a system of linear algebraic equations for the values of functions $\varphi^{(k)}$, $\psi^{(k)}$, $\varphi_n^{(k)}$, $\psi_n^{(k)}$ ($k = 0,1$) at the points $M_i^{(k)}$. The sampled conditions (3) are required to be attached to the system to determine the constants C_0, C_1.
By solving the resulting system, it is possible to determine the unknown boundary values of the required functions, that allow to determine the approximate value of the biharmonic stream function Ψ at any point of the domain D.

The test examples

The viscous fluid flow between two rotating concentric circular cylinders
Let ρ_0 be a radius of an internal cylinder (with angular velocity of ω_0) and ρ_1 be a radius of an external cylinder (with angular velocity of ω_1). The exact solution of this problem is described in [1], for example. In fig. 1 the red solid line shows the plot of the exact absolute value of the velocity depending on ρ, and the blue dashed line is the plot of the corresponding velocity found numerically ($N_0 = 200, N_1 = 400$).

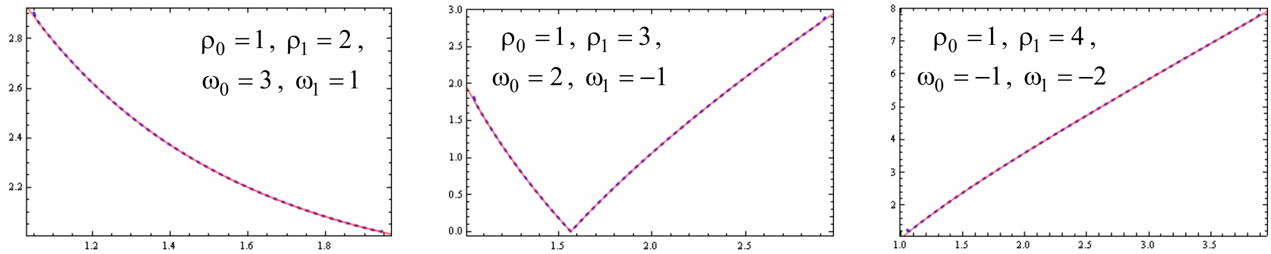
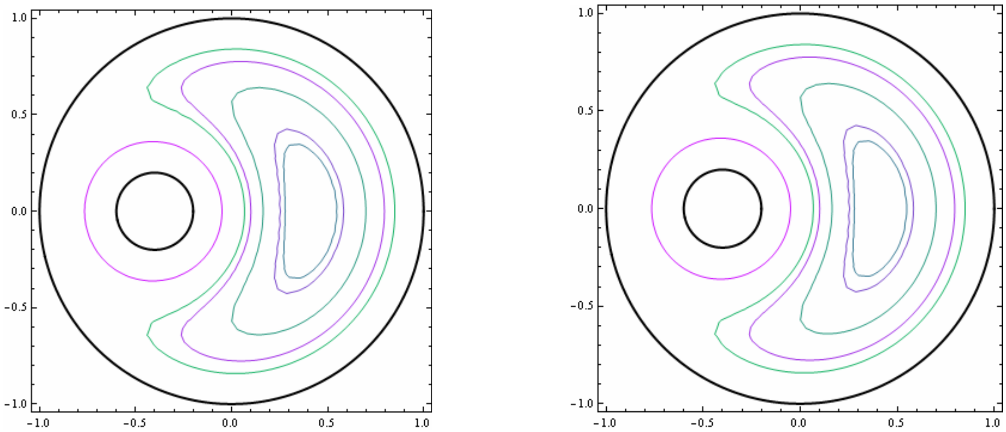


Figure 1. The plots of the velocity of the viscous fluid as function of the polar radius

The viscous fluid flow between two rotating eccentric circular cylinders
Let the cylinders centers be located on the axis Ox, and the distance between them be equal to $\hat{x}$. To obtain an exact analytical solution the problem is considered in the polar coordinates [3], [4]. In fig. 2 it is shown a comparison of the results of the numerical and analytical solutions for $N_0 = 40$, $N_1 = 100$.



(a): the exact solution (b): the numerical solution

Figure 2. The streamlines $\Psi = const$ for $\rho_0 = 0.2$, $\rho_1 = 1$, $\omega_0 = 5$, $\omega_1 = -0.2$, $\hat{x} = 0.4$

Conclusions

It is clear from the presented plots that there is a good agreement between the numerical and exact results inside the domain even for small values of N_0 и N_1. Thus the suggested algorithm for the calculation of the viscous fluid flow between two arbitrarily moving cylinders of an arbitrary cross-section may be considered effective and precise.

References

[1] N.A. Slyozkin, The dynamics of a viscous incompressible fluid. (GITTL, Moscow, 1955) [in Russian].
[2] N.E. Joukowski and S.A. Chaplygin, “Friction of a Lubricating Layer between the End Journal and the Bearing”, Tr. Otdeleniya Fiz. Nauk O-va Lyubitelei Estestvoznaniya 13, No. 1. 24–33 (1906) [in Russian].
[3] B.Y. Ballal and R.S. Rivlin, “Flow of a Newtonian Fluid Between Eccentric Rotating Cylinders: Inertial Effects”, Arch. Rational Mech. Anal. 62, 237-294 (1976).
[4] A.O. Kazakova and A.G. Petrov, “Viscous Fluid Velocity Field between Two Cylinders which Rotate and Move Translationally”, Fluid Dynamics, 2016, Vol. 51, No. 3, pp. 311–320.
[5] Petrov A.G. Quadrature Formulas for Periodic Functions and Their Application to the Boundary Element Method // Computational Mathematics and Mathematical Physics. V. 48. 2008. pp. 1266-1283.