Stochastic sensitivity in dynamic bifurcations with delayed feedback revealed through multiple scales analysis

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Summary. Dynamic bifurcations commonly appear in multi-degree-of-freedom nonlinear systems with slow-fast dynamics. A multiple scale viewpoint of fast dynamics with a slowly varying bifurcation parameter leads to a non-autonomous reduced system. Given few techniques for non-autonomous systems, particularly for systems with delayed feedback, we develop new asymptotic approaches in the context of two applications with delayed feedback. The approaches illuminate stochastic effects and inform insightful computations.

Dynamic bifurcations Dynamic bifurcations (DBs) refer to nonlinear transitions that occur when a bifurcation or control parameter varies. In contrast to a static bifurcation where a change in qualitative behavior corresponds to a fixed parameter value, the dynamic transition occurs as the parameter traverses a range. Often the transition is shifted relative to static values, earning the name delayed bifurcation (not to be confused with delayed feedback). In contexts without delayed feedback [1], it is convenient to view a dynamic bifurcation with multiple time scales, where the bifurcation parameter varies slowly relative to the rest of the fast dynamics. DBs in systems without delayed feedback have been studied analytically using asymptotic analysis and geometric singular perturbation theory, and for stochastic settings with both path-dependent analyses and time-dependent densities. Systems with delayed feedback have special properties, so we can use some of these concepts but not all.

Machine tool dynamics To fix ideas, we discuss a dynamic Hopf bifurcation in the context of a single degree of freedom non-dimensionalized model with delayed feedback for the location z of a machine tool with time delay τ and material parameters in κ_j,

$$z_{tt} + cz_t + z = \kappa_1(\epsilon t) + \kappa_2(\epsilon t)[z(t-\tau) - z(t)], \epsilon \ll 1\,. \tag{1}$$

The static case of $\epsilon = 0$, κ_j = constant has been studied extensively [2], establishing a critical value for a Hopf bifurcation at $\kappa_2 = \kappa_H$. For $\kappa_2 < \kappa_H$ the constant value steady state is stable, while for $\kappa_2 > \kappa_H$, it is unstable and oscillations in z grow, exhibiting chatter in the dynamics. This analysis is accomplished by looking at eigenvalues for the system linearized about the steady state and calculating curves corresponding to vanishing real part of this eigenvalue. The case of $0 < \epsilon \ll 1$ exhibits a DB, where there is a shift in the value $\kappa_2 = \kappa_d$ near which oscillatory behavior grows, i.e. $\kappa_d > \kappa_H$ (see Fig. 1 LEFT). To understand this transition, it is not sufficient to look simply at fixed parameter values where the real part of the eigenvalues are zero, in contrast to the analysis for the constant parameter case.
For systems without delay, the DB mechanism can be related to the integral of the eigenvalue describing growth of the oscillations, rather than to its instantaneous value. We illustrate this for (1). using a multiple time scale analysis for oscillations around the quasi-steady state, z_{slow} depending on a slow time scale $T = \epsilon t$, $\kappa_j(T)$, and other parameters.
We consider the time dependent growth (or decay) of perturbations to z_{slow}, $z(t,T) = z_{\text{slow}} + \epsilon Z(t,T)$ with Z satisfying to leading order (1) with $\kappa_1 = 0$. Taking $Z(t,T) = e^{(r(T)+i\omega(T))/\epsilon} Z_0$ in terms of the slow time T, we get

$$m[r_T^2 - \omega_T^2] + cr_T + k \;=\; \kappa_2(T)\left[e^{\Delta_{\epsilon\tau} r}\cos(\Delta_{\epsilon\tau}\omega) - 1\right] \tag{2}$$
$$2mr_T\omega_T + c\omega_T \;=\; \kappa_2(T)\left[e^{\Delta_{\epsilon\tau} r}\sin(\Delta_{\epsilon\tau}\omega)\right] \tag{3}$$

where $\Delta_{\epsilon\tau}\omega = \frac{\omega(T-\epsilon\tau)-\omega(T)}{\epsilon}$, $\Delta_{\epsilon\tau} r = \frac{r(T-\epsilon\tau)-r(T)}{\epsilon}$. In general this system captures the transition, but comparing the static and dynamic parameter cases requires a new approach beyond (2-3). For the static case, one would normally substitute directly $Z = e^{\lambda t = i\omega t}$. For the dynamic case we use $Z = e^{\lambda(T)}$, $\lambda(T) = \rho(T) + i\psi(T)$. Then the growth in the solution is in terms of $\int \rho_T(T)dT$, where

$$2m\rho_T \sim -(c + \kappa_2(T)\tau(\cos(\Delta_{\epsilon\tau}\psi) - 1)) + \sqrt{(c + \kappa_2(T)\tau(\cos(\Delta_{\epsilon\tau}\psi) - 1))^2 - 4m(k - \kappa_2(T)\left[\cos(\Delta_{\epsilon\tau}\psi) - 1\right] - m\omega_{qs}^2)} \tag{4}$$

with the approximations $\Delta_{\epsilon\tau}\rho(T) \sim \tau\frac{d\rho(T)}{dT}$, $e^{\Delta_{\epsilon\tau}\rho} \sim 1 + \Delta_{\epsilon\tau} r$, and ω_{qs} the quasi-static approximation for the frequency. In contrast, for the static case, $\rho \sim CT$ with C obtained from (4) with κ_2 a constant near κ_H. While the static and dynamic cases rely on similar functional forms, the change in sign of ρ for the static case depends on the instantaneous rate coefficient C in terms of fixed κ_j. In contrast in (4), ρ is determined through accumulation via integration over T, including a range where $\kappa_2(T) < \kappa_H$. This calculation illustrates the importance of understanding the dynamical multiple scales behavior of both the real and imaginary parts of exponents, while in the static case the focus is on the imaginary part and parameters for which $C = 0$. In Figure 1LEFT , we see $\kappa_d > \kappa_H$, κ_d depends on ϵ for smaller values of ϵ, with increased shift in κ_d for smaller initial values of $\kappa_2(T)$ and larger ϵ as observed in other applications [1].

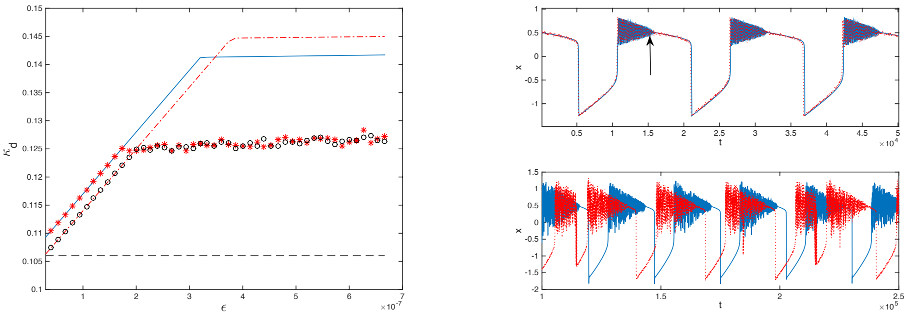

Figure 1: LEFT: The dynamic bifurcation value κ_d vs. ϵ, capturing the growth of oscillatory dynamics. $\kappa_H \approx .106$ (dashed line). Solid and dash-dotted lines: $\kappa_2(T) = \kappa_d$ for which oscillations grow in the deterministic model; dash-dotted corresponds to a smaller initial value of κ_2. $*$ and o indicates κ_d for noise introduced in κ_j. RIGHT: Comparison of dynamics of x in (5) with (red dotted) and without noise (blue solid), in cases with (lower) and without (upper) stochastic sensitivity.

Stochastic sensitivity In addition to the deterministic dynamics for r and ω , we can also consider the stochastic dynamics in the case where there is variability in the material parameters (or other parameters). As illustrated above, it is the integral of r that contributes to the DB. It is well known that via coherence resonance, oscillations can be sustained for material parameters $\kappa_2 < \kappa_H$ [3], so that for the DB, on average, stochastic contributions to the integral of r accumulate and reduce the shift in κ_d. We observe this in Figure 1 LEFT, for parameter values where there is a noticeable shift in κ_d. Note that the noise removes the shift in κ_d for smaller initial values of κ_2, except for very slow time scales T.

Implications for other applications: The example above illustrates the importance of understanding time-varying quantities such as amplitude and frequency of perturbations in the dynamic stability in non-autonomous systems. To contrast with the behavior for machine tool dynamics indicated above, we consider another application, that of an opto-electronic oscillator. We give a non-dimensionalized equation for x a state variable and y, its integral capturing band pass features commonly appearing in optical communications [4],

$$\epsilon x' = -x - y + f(x(s-\delta)) - f(0), \;\; y' = x, f(x) = \frac{\beta}{1 + m\sin^2(x+\phi)} \qquad \epsilon \ll 1\,. \tag{5}$$

A physically relevant range of parameters corresponds to $\epsilon \ll \delta \ll 1$, allowing a multiple time scales analysis based on a slow manifold $(x_0(s), y_0(s))$ obtained for $\epsilon = 0$. Studying perturbations u to the slow manifold, i.e. $x = x_0 + u$, x_0 is treated as a quasi-steady state relative to a fast time s/ϵ. We find $x_0 = x_H$ corresponding to a Hopf bifurcation in u, so that x_0 plays the role of the DB parameter. As u oscillates around the upper branch of slow manifold (see Figure 1 RIGHT) these oscillations lose stability via a Hopf-type DB, shifted relative to x_H marked by an arrow in the figure.
The multiple scale analysis of x_0 and u leads to an understanding of the stochastic sensitivity of this system. We derive an expression $\sigma(1+\epsilon(\sigma^2+\omega^2)) = -1 - e^{-\delta\sigma/\epsilon}\cos(\delta\omega/\epsilon) m f(x_0)\sin(2(x_0+\phi))$ where σ characterizes the growth or decay of noisy perturbations near $x = 0 = x_H$, related to oscillations in u with frequency ω. Near x_H, σ_{x_0} is large. Then, unlike the machine tool model above, there is not a substantial region $|\sigma| \ll 1$ where a coherence resonance-like pheomena can change the DB. Rather, even though there is a shift of the DB relative to x_H, noisy perturbations can not sustain oscillations via coherence resonance so the effect of generic noise is relatively minimal, and does not affect the DB. The upper panel of Figure 1RIGHT shows this, where the deterministic and stochastic dynamics are nearly identical. However, noise can have a noticeable effect, via perturbations of the larger amplitude of oscillations away from the Hopf point. There noise can drive crossings of the unstable portion of the solution of (5) for $\epsilon = 0$, rather than via coherence resonance as in the machine tool example above.

Conclusions: In two examples of dynamic bifurcation in systems with delayed feedback, reduced non-autonomous systems are derived via a multiple scales analysis. The stability analyses borrow from standard techniques for systems with delay, with new sophisticated approaches for transitions exploiting variations in real and imaginary parts of exponents for fluctuations about quasi-steady states. We identify mechanisms both for shifts in the transitions and for stochastic sensitivity, opening the door for studying hysteresis between steady states and complex nonlinear delayed dynamics as well as the robustness of transitions, or tipping, in non-autonomous settings with delayed feedback. Related asymptotics are useful for large delays $\tau = O(\epsilon^{-1})$, of interest in the Ikeda family of optoelectronic oscillators [5].

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