Symmetry-induced dynamic localization in lattice structures

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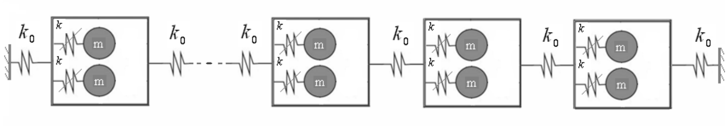
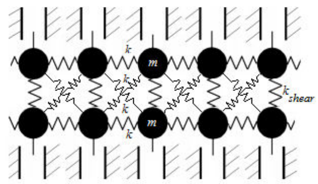
Summary. Internal symmetry in lattice structures may give rise to stationary compact solutions, even in the absence of disorder or nonlinearity. These compact solutions are related to the existence of flat dispersion curves (bands). Nonlinearity can be a destabilizing factor for such compactons.

Mechanical models and dispersion analysis

Two mechanical systems in which configurational symmetry results in existence of a flat dispersion curve (band), are presented in Fig. 1: a system of massless boxes connected by linear springs with two identical oscillators in each box in Fig. 1(a), and a system of two identical uniform cross-linked chains with vibro-impact on-site potential in Fig 1(b).



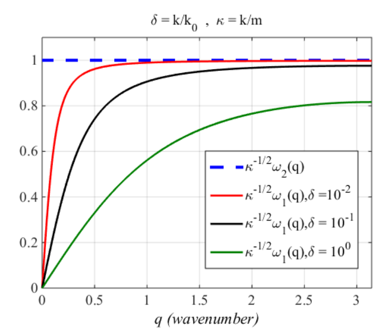
(a)



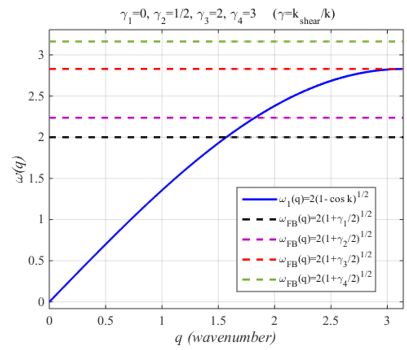
(b)

Fig.1 Mechanical systems having flat dispersion bands

In the linear regime, where the in-box potential in the system in Fig.1(a) is harmonic, and the displacements of the masses in the system of close cross-linked chains in Fig. 1(b) are small enough to avoid impacts, dispersion analysis shows that both systems possess a flat dispersion band, as demonstrated in Fig. 2.



(a)



(b)

Fig.2 Dispersion bands (a) of the system in Fig. 1(a) and (b) of the system in Fig. 1(b) with the flat bands drawn in dashed lines.

As noted in [1], modes corresponding to the flat bands are spatially detached, or perfectly compact. For the system in Fig. 1(a) it is anti-synchronous motion of the two oscillators in just on box, with other masses at rest, and experiencing zero net force from the box with the internal motion. For the system in Fig. 1(b), it is anti-synchronous motion of one mass of the upper chain and the mass just below it in the lower chain, in which case zero net force is exerted on all other masses if there are at rest and positioned uniformly.

Nonlinear regime

In the nonlinear regime, compact solutions may still exist, yet their stability may be lost, depending on parameters.

First system
The chain of massless boxes given in Fig. 1(a) when the in-box linear interaction forces are augmented by cubic terms, still admits a compact solution, if compact anti-symmetric initial conditions are assumed, due to the oddness of the cubic function. If, however, perturbation to antisymmetric compact initial conditions is introduced, it may or may not grow. Linear stability is performed rigorously for the case of a single box-on-a-spring. The solution for the antisymmetric mode (distance between two in-box masses) in this case is given by the 5th elliptic Jacobi function 'sd',

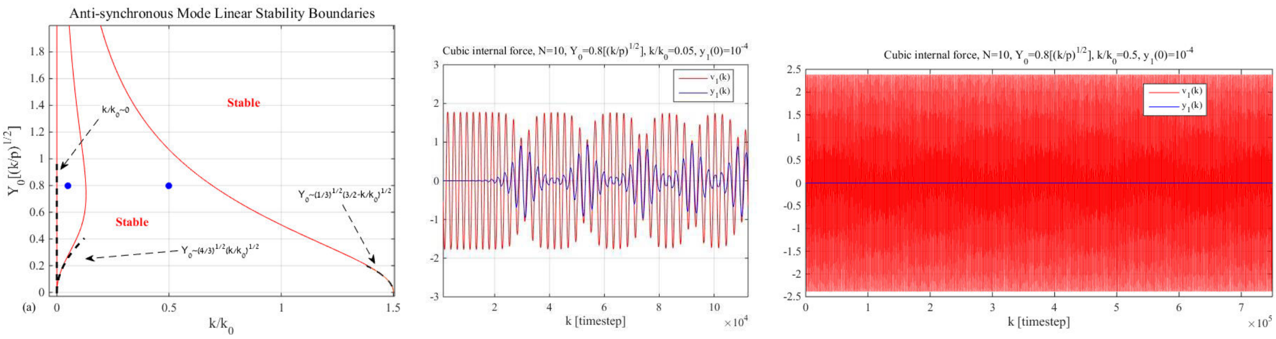
$$v(t) = Y_0\sqrt{4C}(1+4\hat{\varepsilon}C)^{-1/4}\mathrm{sd}\left\{(1+4\hat{\varepsilon}C)^{1/4}\omega t \,\middle|\, (1/2)[1-(1+4\hat{\varepsilon}C)^{-1/2}]\right\} \tag{1}$$

where $$C = 4\hat{\eta}\hat{\varepsilon}^2 + (1+4\hat{\eta})\hat{\varepsilon} + 1 + \hat{\eta}, \hat{\eta} \triangleq 2k/k_0, \hat{\varepsilon} = pY_0^2/(2k), \omega \triangleq \sqrt{k/m} \quad (2)$$

and where p is the cubic force term pre-factor and Y_0 is the amplitude of synchronous motion having the same energy as the anti-synchronous one. The linear stability problem of the anti-synchronous solution is given by a Hill equation, where the Hill function is given by the Fourier series of the following function:

$$h(t) \triangleq [1+(3/2)\hat{\varepsilon}v^2(t)] / \{1+[\hat{\eta}/(1+\hat{\eta})](3/2)\hat{\varepsilon}v^2(t)\} \quad (3)$$

Since the 5th Jacobi function has a known Fourier series, the function $h(t)$ can be Fourier-decomposed using first analytical convolution and then numerical deconvolution by solution of a linear system, as done in [2]. Hill analysis then shows the emergence of an instability tongue, with stable surroundings, as given in Fig.3(a). Consecutive numerical integration of a finite chain (of N=10 boxes) with the cubic internal force augmentation, taking parameters both from the unstable tongue and the stable surroundings, as calculated for a single box, reveals the existence of stable compact solutions in the considered nonlinear chain. Illustration of unstable/stable compactons in a chain is given in Fig. 3(b-c).



(a) (b) (c)

Fig.3 Stability diagram for N=1 (a), anti-synchronous mode history in the box with initial compacton for N=10 for k/k_0 in the N=1-instability tongue (k_0=20k) (b), and stable compacton for k_0=2k (c). $y_i(t)$ is relative centre-of-mass displacement of the i^{th} box.

Second system

When the displacements of the masses in the cross-linked chain depicted in Fig. 1(b) are large enough, impacts occur, rendering the system non-smoothly nonlinear. A perfectly compact anti-synchronous periodic solution still exists, since sharp impacts only change the signs of the velocities of two masses, which happens simultaneously, and thus the positions stay antisymmetric at all times leading to external force cancellation for the compacton. This impacting solution takes the form:

$$v_1(t) = \pi S_0^{-1}\left|\pi - 2\pi\{\omega t/(2\pi)\}_{frac}\right|, \; S_0 \triangleq \sum_{n=1,3,5}^{\infty}\left[(\omega^2 n^2-4)^2-16\right]^{-1/2} \quad (4)$$

where ω is now the frequency of the periodic motion, measured in units of $(k/m)^{1/2}$ and $\{\}_{frac}$ is the fractional part of a number. The stability of this solution is examined by Floquet theory for N>>1. The monodromy matrix is constructed analytically, using the notion of a saltation matrix, applying the approach as in [3], twice consequently, once for each impact. In the small k_{shear}/k limit, where asymptotic analysis of a single element produces a good estimate, numerical stability analysis reveals the existence of a stable region in the frequency – stiffness ratio plane, as shown in Fig. 4.

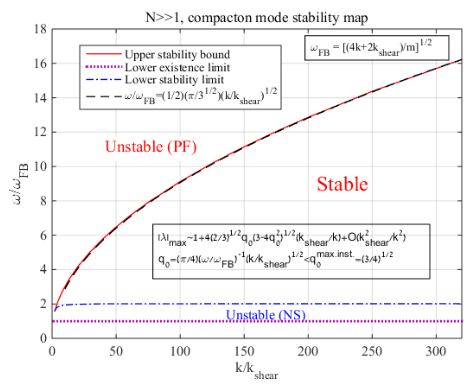


Fig.4 Stability map showing the existence of stable compactons for intermediate frequencies, along with an asymptotic estimate.

[1] Flach, S. et al., Detangling flat bands into Fano lattices EPL, 105 (2014) 30001
[2] Perchikov, N., Gendelman, O.V., Nonlinear dynamics of hidden modes in a system with internal symmetry, J. Sound. Vib. 377 (2016) 185-215
[3] Bernardo, Mario di, et al. Piecewise-smooth dynamical systems: theory and applications. Vol. 163. Springer Science & Business Media, 2008