Modular Approach for the Modeling and Dynamic Analysis of a Pipe Conveying Fluid

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Summary. In this paper, a modular methodology, in which constraints associated to compatibility conditions can be included *a posteriori*, is applied for the derivation of a nonlinear reduced order model of a cantilevered pipe conveying fluid, a classical problem in the area of Fluid-Structure Interaction (FSI). The dynamic response of this model is detailed in the neighborhood of an equilibrium reference state, by describing the dependency of the eigenvalues of its linearized form with respect to its parameters. The assessment of the numerical analysis includes comparisons with benchmark results, which are reproduced accurately, indicating that the adopted modeling approach might be successfully applied for other problems in FSI.

Introduction

This paper is part of a series of publications in which a novel approach for the modeling of Fluid-Structure Interaction systems is introduced and discussed. This new methodology allows to explore the modularities and symmetries within a given dynamic system, providing an algorithm in which all the effects due to kinematic constraints can be computed *a posteriori*, using adequate projection operators [4, 5].

Consider the benchmark problem of a pipe conveying fluid to which Païdoussis [7] devoted a comprehensive treatise, with an extensive literature review. In a first approximation, the pipe can be considered as a homogeneous hollow solid cylinder that satisfies the Kirchhoff-Love beam model (small diameter when compared to the length), with an internal axial incompressible plug-flow (in which the boundary layer adjacent to the inner wall is neglected). It can be stated that the only source of nonlinearities are the large displacements that the pipe is allowed to perform. Once the compatibility conditions are motion constraints, their effect in the dynamics of the system can be included *a posteriori*.

Orsino and Pesce [3] already applied a recursive form of the modular approach for obtaining a FEM model for this system, by proposing a 4-level hierarchy of equations of motion (Figure 1) in which only the models at levels 2 and 3 satisfy both boundary and compatibility conditions. The simplicity brought to the modeling procedure due to the possibility of computing the nonlinear effects *a posteriori* and the successful qualitative results obtained in numerical simulations, motivates further investigation on how this novel approach can be used to describe comprehensively the dynamic response of this system in the neighborhood of a reference state. This can be done by linearizing the system and analyzing the dependency of its eigenvalues with respect to its parameters, which allows to identify the critical values of parameters for which transitions to instability are observed. The scope of this paper is to propose a strategy to perform these analyses, based on the modular approach, and to compare the results so obtained with benchmark ones, available in the literature.

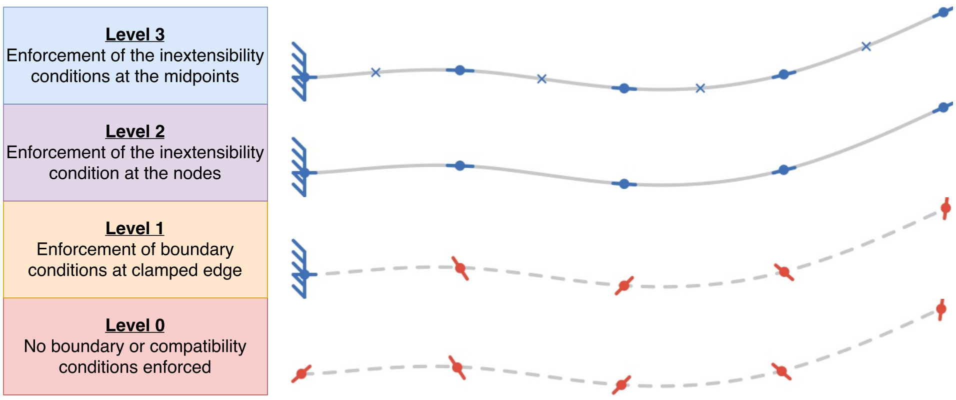


Figure 1: 4-level hierarchy representing the FEM modeling a pipe conveying fluid adopted in [3]. The motion of the red nodes do not respect the boundary or compatibility conditions; the motion of the blue nodes (and midpoints) do.

Although following the same general discrete formulation presented in [3] in which the compatibility conditions are only enforced *a posteriori*, the modeling strategy adopted in this paper will use modal-like projection functions in the Galerkin discretization instead of the Finite Element Method (FEM). Doing so, not only the boundary conditions can be identically satisfied, making it unnecessary to enforce these constraints *a posteriori*, but also it is expected to recover results of the benchmark linear distributed parameter models with a low DOF reduced order one. As it will be discussed later, the use of a reduced order model allows to enforce the compatibility conditions at a finite number of points only (in the one-dimensional model, each point represents a cross section of the pipe). It is expected, however, that due to the smoothness of the adopted projection functions, the violation of these constraints at any other point is a second order effect. This is a reasonable hypothesis for the derivation of a linearized model in which only small displacements in the neighborhood of a reference configuration are considered; when it comes to a nonlinear model for large displacements, the violation of the compatibility conditions should be monitored along the corresponding simulations. The 2-level hierarchy of equations of motion to be used in the modeling of a cantilevered pipe conveying fluid shown in this paper is illustrated in Figure 2.

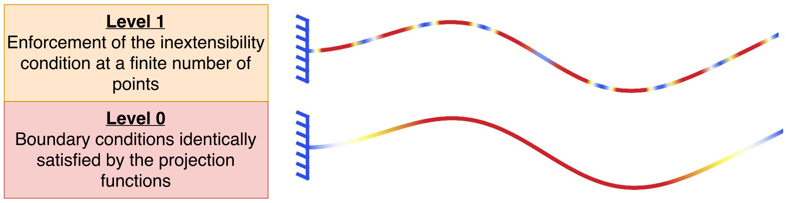


Figure 2: 2-level hierarchy representing the modular modeling of a cantilevered pipe conveying fluid developed in this text. Color scale indicates regions close to the points in which the constraints are enforced in blue.

Taking advantage of a mass matrix that is proportional to the identity and considering that the constraint are enforced only once in the 2-level hierarchy proposed, which means that it is not necessary a recursive algorithm, the approach used for obtaining the equations of motion corresponding to level 1, given the ones at level 0 and the constraint equations associated to the inextensibility condition, is based on the use of Udwadia-Kalaba equation [8, 9]. Orsino [4] proves that such an approach is equivalent to a non-recursive form of a generic constraint enforcement algorithm based on the modular methodology.

After obtaining the nonlinear equations of motion of the cantilevered pipe conveying fluid, it is necessary to perform a linearization procedure in order to be able to analyze its dynamic response for small motion in the neighborhood of the reference state and to make quantitative comparisons with the results associated to benchmark linear models. Following [7, 10], in nondimensional form, the linear model depends on three parameters: the nondimensional speed of the internal flow with respect to the pipe (u), the ratio of the mass per unit of length of the fluid to the total mass per unit of length (β) and a nondimensional gravitational field (γ). Following the linearization procedures proposed in [5], an algorithm can be used for evaluating the eigenvalues of the system as a function of these parameters, outlining a description of its dynamic response. A similar approach has already been applied in [5] in the analysis of a tadpole tricycle.

This paper is divided in four sections, the first being this introduction and the last being reserved for conclusions. In the second section the general discrete formulation of the problem developed in [3] is revisited and specialized according to obtain a reduced order model based on modal-like projection functions. It is discussed how Udwadia-Kalaba equation can be applied to enforce inextensibility constraints in the model, and how the linearization algorithms can applied to it. The third section presents some numerical results, in which the eigenvalues of the linearized model are analyzed as a function of the parameters u, β and γ, the results are compared with the benchmark ones by [2, 7], and some further discussions are presented.

Finally, it is useful to make some remarks concerning the notation adopted in this text. Column-matrices are considered to be equivalent to tuples and are represented by bold lowercase letters. The bold uppercase notation is reserved to matrices that can only be represented by two-dimensional arrays. Also, let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary column-matrices; the operators "$\cdot$" and "$\otimes$" are defined as follows:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y} \qquad \text{and} \qquad \mathbf{x} \otimes \mathbf{y} = \mathbf{x}\mathbf{y}^{\mathrm{T}} \tag{1}$$

Modular modeling of an inextensible cantilevered pipe conveying fluid

The modeling of an inextensible cantilevered pipe conveying fluid detailed in this section follows the same hypotheses and the same general discrete formulation already presented in [3]. Particularly, the hypotheses also correspond to the ones adopted in the derivation of the classical benchmark models presented by Païdoussis [7]. The pipe is an homogeneous hollow cylinder of linear elastic material whose nondeformed configuration corresponds to a straight centerline, which is assumed to aligned to the local gravitational field whenever its effects are non-negligible. The model is two-dimensional, so that out-of-plane motions are not considered. There is an internal steady axial flow from the clamped end to the free end (where there is a discharge), which is assumed to be a plug-flow of constant speed U with respect to the pipe. No external flow effects are considered. An arc-length coordinate s is defined along the center line, with $s = 0$ corresponding to the clamped end and $s = \ell$ to the free end. Define axes x and z so that the former coincides with the centerline in the nondeformed configuration and is oriented from the clamped end (origin) to the free end, and the latter also lies in the plane of motion, being orthogonal to the first. Considering that $\ell \gg d_e$, with d_e being the external diameter of the pipe, it can be assumed that the description of the coordinates x and z of each point of the centerline is sufficient for uniquely defining each admissible configuration of this system. Particularly, a Galerkin discretization is adopted, so that there are column-matrices of projection functions, $\mathbf{n}_x(s)$ and $\mathbf{n}_z(s)$, and column-matrices of generalized coordinates, $\mathbf{q}_x$ and $\mathbf{q}_z$, such that x and z coordinates of the points in the centerline of the pipe can be approximated modeled by the following expressions:

$$x = s + \mathbf{n}_x(s) \cdot \mathbf{q}_x \quad \text{and} \quad z = \mathbf{n}_z(s) \cdot \mathbf{q}_z \tag{2}$$

Following the Hamiltonian derivation presented in [3], the equations of motion obtained for the inextensible pipe if neither compatibility nor boundary conditions are considered a priori are:

$$\mathbf{M}_j\ddot{\mathbf{q}}_j + (\mathbf{B}_j - \mathbf{B}_j^{\mathrm{T}} + \mathbf{E}_j)U\dot{\mathbf{q}}_j + ((\mathbf{W}_j + \mathbf{F}_j)U^2 + \mathbf{K}_j)\mathbf{q}_j = \mathbf{g}_j \quad \text{for} \quad j = x, z \tag{3}$$

Coefficient matrices $\mathbf{M}_j$, $\mathbf{B}_j$ and $\mathbf{W}_j$, $j = x, z$, are associated to the inertial forces and are defined as follows:

$$\mathbf{M}_j = \int_0^\ell (m_{\mathrm{p}} + m_{\mathrm{f}})(\mathbf{n}_j \otimes \mathbf{n}_j)\,\mathrm{d}s, \quad \mathbf{B}_j = -\int_0^\ell m_{\mathrm{f}}(\mathbf{n}_j' \otimes \mathbf{n}_j)\,\mathrm{d}s \quad \text{and} \quad \mathbf{W}_j = -\int_0^\ell m_{\mathrm{f}}(\mathbf{n}_j' \otimes \mathbf{n}_j')\,\mathrm{d}s \tag{4}$$

with m_{p} and m_{f} standing for the masses of the pipe and the fluid per unit of length of the pipe. Denoting by EI the flexural rigidity of the pipe and by g the magnitude of the local gravitational field, the stiffness matrices $\mathbf{K}_j$ and the generalized gravitational forces column-matrices $\mathbf{g}_j$, $j = x, z$, are given by the following expressions:

$$\mathbf{K}_j = \int_0^\ell EI(\mathbf{n}_j'' \otimes \mathbf{n}_j'')\,\mathrm{d}s \tag{5}$$

$$\mathbf{g}_x = \int_0^\ell (m_{\mathrm{p}} + m_{\mathrm{f}})g\mathbf{n}_x\,\mathrm{d}s \quad \text{and} \quad \mathbf{g}_z = 0 \tag{6}$$

Finally, the terms associated to the effects of the flow discharge lead to the definition of the coefficient matrices $\mathbf{E}_j$ and $\mathbf{F}_j$, $j = x, z$:

$$\mathbf{E}_j = (m_{\mathrm{f}}\mathbf{n}_j \otimes \mathbf{n}_j)\Big|_{s=\ell} \quad \text{and} \quad \mathbf{F}_j = (m_{\mathrm{f}}\mathbf{n}_j \otimes \mathbf{n}_j')\Big|_{s=\ell} \tag{7}$$

One of the objectives of this work is to check if the model obtained by the modular methodology is able to reproduce the dynamic response of the benchmark linear one by Gregory and Païdoussis [2]. Thus, instead of adopting interpolating polynomials as projection functions, which would lead to a FEM model similar to the one obtained in [3], the Galerkin discretization will be based on the use of modal-like functions. The set of functions corresponding to the modes of a cantilevered Euler-Bernoulli beam, for example, represents a linear independent set of functions identically satisfying the boundary conditions of the problem addressed in this section. It means that any mode of the linearized cantilevered pipe can be represented by a convergent series of functions of this first set; a truncated series (i.e. a linear combination of the first n functions of the first set, for a finite n) would correspond to an approximate representation.
A function representing the m-th mode of a cantilevered Euler-Bernoulli beam with respect to the nondimensional arc-length variable $\xi = s/\ell$ is:

$$\phi_m(\xi) = \cosh(\lambda_m\xi) - \cos(\lambda_m\xi) - \sigma_m\left(\sinh(\lambda_m\xi) - \sin(\lambda_m\xi)\right) \qquad \text{with} \qquad \sigma_m = \frac{\sinh(\lambda_m) - \sin(\lambda_m)}{\cosh(\lambda_m) - \cos(\lambda_m)} \tag{8}$$

The boundary conditions are $\phi_m(0) = \phi_m'(0) = 0$, at the clamped end, and $\phi_m''(1) = \phi_m'''(1) = 0$, at the free end. Note that apart from $\phi_m''(1) = 0$, the other three boundary conditions are identically satisfied by the function given by equation (8); in order to ensure that $\phi_m''(1) = 0$, λ_m must be the m-th positive root of the equation $\cosh\lambda\cos\lambda = -1$. It should also be noticed that $\{\phi_1(\xi), \ldots, \phi_m(\xi), \ldots\}$ is a set of orthonormal functions [1], i.e.:

$$\int_0^1 \phi_i(\xi)\phi_j(\xi)\,\mathrm{d}\xi = \delta_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \tag{9}$$

Thus, if one takes $\mathbf{n}_x(s) = \mathbf{n}_z(s) = (\phi_1(s/\ell), \ldots, \phi_n(s/\ell))$ for the Galerkin discretization both in x and z directions, see equation (2), the boundary conditions of the problem will be identically satisfied, regardless the values of the generalized coordinates $\mathbf{q}_x$ and $\mathbf{q}_z$. Therefore, the use of such an strategy of discretization, in which the projection functions ensure the satisfaction of the boundary conditions, reduces the application of the modular methodology in this problem to the *a posteriori* enforcement of compatibility conditions to the "decoupled" system of equations of motion (3). Under the assumed hypotheses, the compatibility condition is associated to the inextensibility, which be expressed by the following identity:

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = 1 \tag{10}$$

To impose inextensibility to this discretized problem, the following conditions must be satisfied:

$$(1 + \mathbf{n}_x'(s) \cdot \mathbf{q}_x)^2 + (\mathbf{n}_z'(s) \cdot \mathbf{q}_z)^2 = 1 \tag{11}$$

$$(\mathbf{n}_x'(s) + (\mathbf{n}_x'(s) \otimes \mathbf{n}_x'(s))\mathbf{q}_x) \cdot \delta\mathbf{q}_x + ((\mathbf{n}_z'(s) \otimes \mathbf{n}_z'(s))\mathbf{q}_z) \cdot \delta\mathbf{q}_z = 0 \tag{12}$$

For the particular choice of projection functions adopted in this problem, n linearly independent constraint equations can be defined based on conditions (11, 12). A strategy for doing so is to enforce these conditions at n points at the centerline of the pipe, identified by the arc-length coordinates $s_1, \ldots, s_n$. Thus, the constraint equations to be used for the *a posteriori* enforcement of inextensibility condition can be expressed in the following matrix form:

$$\mathbf{A}\begin{bmatrix}\ddot{\mathbf{q}}_x\\ \ddot{\mathbf{q}}_z\end{bmatrix} = \mathbf{b} \tag{13}$$

with $\mathbf{A}$ and $\mathbf{b}$ given by:

$$\mathbf{A} = \begin{bmatrix}(\mathbf{n}'_x(s_1) + (\mathbf{n}'_x(s_1)\otimes\mathbf{n}'_x(s_1))\mathbf{q}_x)^{\mathrm{T}} & ((\mathbf{n}'_z(s_1)\otimes\mathbf{n}'_z(s_1))\mathbf{q}_z)^{\mathrm{T}}\\ \vdots & \vdots\\ (\mathbf{n}'_x(s_n) + (\mathbf{n}'_x(s_n)\otimes\mathbf{n}'_x(s_n))\mathbf{q}_x)^{\mathrm{T}} & ((\mathbf{n}'_z(s_n)\otimes\mathbf{n}'_z(s_n))\mathbf{q}_z)^{\mathrm{T}}\end{bmatrix} \tag{14}$$

$$\mathbf{b} = \begin{bmatrix}-\dot{\mathbf{q}}_x\cdot((\mathbf{n}'_x(s_1)\otimes\mathbf{n}'_x(s_1))\dot{\mathbf{q}}_x) - \dot{\mathbf{q}}_z\cdot((\mathbf{n}'_z(s_1)\otimes\mathbf{n}'_z(s_1))\dot{\mathbf{q}}_z)\\ \vdots\\ -\dot{\mathbf{q}}_x\cdot((\mathbf{n}'_x(s_n)\otimes\mathbf{n}'_x(s_n))\dot{\mathbf{q}}_x) - \dot{\mathbf{q}}_z\cdot((\mathbf{n}'_z(s_n)\otimes\mathbf{n}'_z(s_n))\dot{\mathbf{q}}_z)\end{bmatrix} \tag{15}$$

Differently from the formulation applied for obtaining the FEM discretized model in [3], in the approach adopted in this text, only the inextensibility condition needs to be enforced *a posteriori*, once the boundary conditions are identically satisfied due to the chosen projection functions. Thus, the most natural approach for the *a posteriori* constraint enforcement is to perform it at once (in a non-recursive way). Also, it can be noticed that due to the property (9), the generalized inertia matrix associated to the "decoupled" system of equations of motion (3) is proportional to the identity matrix once, from (4), $\mathbf{M} = \mathrm{diag}(\mathbf{M}_x, \mathbf{M}_z) = (m_\mathrm{p} + m_\mathrm{f})\ell\,\mathbf{1}$. Therefore, in this particular formulation, the application of Udwadia-Kalaba equation [8, 9] is a good alternative for a non-recursive constraint enforcement. Denote by $\mathbf{a}$ and $\ddot{\mathbf{q}}$ the column-matrices constituted by the values of $(\ddot{\mathbf{q}}_x, \ddot{\mathbf{q}}_z)$ that satisfy equations (3) and that represent the actual values of the generalized accelerations of the system (compatible with both boundary and compatibility constraints), respectively. According to Udwadia-Kalaba equation it can be stated that:

$$\ddot{\mathbf{q}} = \mathbf{a} + \mathbf{M}^{-1/2}(\mathbf{A}\mathbf{M}^{-1/2})^{+}(\mathbf{b} - \mathbf{A}\mathbf{a}) \tag{16}$$

with $(\cdot)^{+}$ denoting the Moore-Penrose generalized inverse of the corresponding matrix. Particularly, once $\mathbf{M}$ is proportional to the identity matrix, (16) can be reduced to the following form:

$$\ddot{\mathbf{q}} = \mathbf{a} + \mathbf{A}^{+}(\mathbf{b} - \mathbf{A}\mathbf{a}) \tag{17}$$

From equation (3), it can be noticed that the $\mathbf{a}$ can be expressed as an affine function of the state $\mathbf{x} = (\mathbf{q}_x, \mathbf{q}_z, \dot{\mathbf{q}}_x, \dot{\mathbf{q}}_z)$ of the system: there are linear terms corresponding to the inertial forces associated to the internal flow and to the elastic forces associated to the flexural rigidity, and there is a constant term associated to the gravitational forces. Also from equation (14), it can also be stated that $\mathbf{A}$ is an affine function of the state variables. Denote by $\mathscr{L}[\cdot]$ the first-order Taylor expansion of a given expression in the neighborhood of the reference state, which in this problem is given by $\bar{\mathbf{x}} = (\bar{\mathbf{q}}_x, \bar{\mathbf{q}}_z, \dot{\bar{\mathbf{q}}}_x, \dot{\bar{\mathbf{q}}}_z) = \mathbf{0}$. Once $\mathscr{L}[\mathbf{a}] = \mathbf{a}$ and $\mathscr{L}[\mathbf{A}] = \mathbf{A}$, it can be stated that:

$$\mathscr{L}[\ddot{\mathbf{q}}] = \mathscr{L}[\mathbf{a} + \mathbf{A}^{+}(\mathbf{b} - \mathbf{A}\mathbf{a})] = \mathbf{a} + \mathscr{L}[\mathscr{L}[\mathbf{A}^{+}](\mathscr{L}[\mathbf{b}] - \mathscr{L}[\mathbf{A}\mathbf{a}])] \tag{18}$$

$\mathscr{L}[\mathbf{b}]$ can be obtained straightforwardly by a first-order Taylor expansion applied to the expression for $\mathbf{b}$ given by equation (15). Also, whenever $\mathscr{L}[\cdot]$ is applied to a product of two terms which are already affine with respect to the state variables, it is enough to cancel second order terms of this product. Therefore, in order to obtain a linearized form of the equations of motion of the reduced order model proposed in this section in the neighborhood of the reference state $\bar{\mathbf{x}} = (\bar{\mathbf{q}}_x, \bar{\mathbf{q}}_z, \dot{\bar{\mathbf{q}}}_x, \dot{\bar{\mathbf{q}}}_z) = \mathbf{0}$, the only non-straightforward procedure is to obtain $\mathscr{L}[\mathbf{A}^{+}]$ without needing the explicit nonlinear expression of $\mathbf{A}^{+}$. The strategy for doing so is detailed in Proposition 4.4 of [5]. Assume that $\mathbf{A} = \bar{\mathbf{A}} + \sum_i x_i\hat{\mathbf{A}}_i$ with $\mathbf{x} = (x_1, x_2, \ldots, x_{4n})$ representing the state vector of the system. Notice that $\bar{\mathbf{A}}$ and all the $\hat{\mathbf{A}}_i$ are constant matrices. The following central finite difference method is used to estimate, with an $\mathscr{O}(h^6)$ accuracy, the partial derivatives of $\mathbf{A}^{+}$ with respect to the state variables:

$$\left.\frac{\partial\mathbf{A}^{+}}{\partial x_i}\right|_{\mathbf{x}=\mathbf{0}} = \sum_{j=1}^{3}\frac{d_j}{h}\left((\bar{\mathbf{A}} + jh\hat{\mathbf{A}}_i)^{+} - (\bar{\mathbf{A}} - jh\hat{\mathbf{A}}_i)^{+}\right) \tag{19}$$

with $d_1 = 3/4$, $d_2 = -3/20$ and $d_3 = 1/60$. Therefore, $\mathscr{L}[\mathbf{A}^{+}]$ can be given by the following expression:

$$\mathscr{L}[\mathbf{A}^{+}] = (\bar{\mathbf{A}})^{+} + \sum_{i=1}^{4n} x_i \left.\frac{\partial\mathbf{A}^{+}}{\partial x_i}\right|_{\mathbf{x}=\mathbf{0}} \tag{20}$$

Once the reference state corresponds to an equilibrium of the system, it can be noticed that $\mathscr{L}[\ddot{\mathbf{q}}]$ must consist of purely linear terms, i.e., its value in $\bar{\mathbf{x}} = \mathbf{0}$ must be identically zero. This condition provides a test for the accuracy of the numerical algorithm used for obtaining the expression of $\mathscr{L}[\mathbf{A}^{+}]$.

Numerical analysis of the dynamic response of the model

The numerical analysis presented in this section is based on the linearized form of the discretized model previously obtained. As already mentioned, the reference equilibrium state is $\bar{\mathbf{x}} = \mathbf{0}$, which corresponds to the pipe at rest in a vertical configuration (centerline is straight and parallel to the local gravitational field). In state space representation, the linearized model can be represented in the following form:

$$\dot{\mathbf{x}} = \left(\dot{\mathbf{q}}, \mathscr{L}[\ddot{\mathbf{q}}](u, \beta, \gamma)\right) = \mathbf{G}(u, \beta, \gamma)\mathbf{x} \tag{21}$$

with $\mathbf{G}$ being a coefficient matrix, independent of the state of the system, which can be expressed as a function of the constant parameters u, β and γ. Basically the remaining of this section will consist of analyzing scenarios in which two of these parameters are fixed and the dependency of the eigenvalues λ of $\mathbf{G}$ on the third parameter is described. Such an approach allows to outline the dynamic response of this system in the neighborhood of the equilibrium state $\bar{\mathbf{x}} = \mathbf{0}$ for these scenarios, which include the identification of bifurcations.

Figure 3 shows $\mathrm{Im}(\lambda)$ versus $\mathrm{Re}(\lambda)$, $\mathrm{Re}(\lambda)$ versus u, and $\mathrm{Im}(\lambda)$ versus u plots for two scenarios in which β assumes two fixed values (0.2 and 0.295, respectively) and the gravitational effects are considered negligible (i.e. $\gamma = 0$); the dependency of the dynamic response of the pipe on the nondimensional internal flow speed is outlined for $0 \leq u \leq 25$, using proper color scales for each plot. These two scenarios are the ones for which $\mathrm{Im}(\lambda)$ versus $\mathrm{Re}(\lambda)$ plots are presented in [2, 7]. As detailed below, these benchmark results correspond to the black dots superimposed to the plots shown in this figure. Figure 4 shows $\mathrm{Im}(\lambda)$ versus u plots (with $0 \leq u \leq 25$) for 6 scenarios in which the values of β and γ are fixed, with a color scale indicating transitions to instability (i.e. sign changes in $\mathrm{Re}(\lambda)$). Figure 5 shows $\mathrm{Im}(\lambda)$ versus β plots for 6 scenarios in which the values of u and γ are fixed, using the same color scale. In these cases, the dependency of the dynamic response on the nondimensional mass ratio β is investigated along the whole physically meaningful interval for this parameter, i.e. $0 \leq \beta \leq 1$.

In order to validate the analysis presented in this section and to make an assessment of the discretized model derived by the modular modeling approach, the results must be compared to ones already available in the literature. The benchmark model for lateral motions of a vertical cantilevered pipe conveying fluid [2, 7, 10] can be represented, in nondimensional form, by the following linear partial differential equation (PDE):

$$\frac{\partial^4 \eta}{\partial \xi^4} + \left(u^2 - \gamma(1-\xi)\right)\frac{\partial^2 \eta}{\partial \xi^2} + 2\beta^{\frac{1}{2}} u \frac{\partial^2 \eta}{\partial \xi \partial \tau} + \gamma \frac{\partial \eta}{\partial \xi} + \frac{\partial^2 \eta}{\partial \tau^2} = 0 \tag{22}$$

In this model, $\eta = \eta(\tau, \xi)$ stands for the nondimensional in-plane lateral displacements of the points in the centerline of the pipe, ξ is a nondimensional length coordinate defined along this centerline, τ is a nondimensional time variable and u, β and γ are the three dimensionless parameters of the problem, which are assumed to be constants. These nondimensional quantities can be related to the variables and parameters of the model presented in the previous section by the following relations:

$$\eta = \frac{z}{\ell}, \quad \xi = \frac{s}{\ell}, \quad \tau = \left(\frac{EI}{m_\mathrm{f} + m_\mathrm{p}}\right)^{\frac{1}{2}} \frac{t}{\ell^2} \tag{23}$$

$$u = U\ell\left(\frac{m_\mathrm{f}}{EI}\right)^{\frac{1}{2}}, \quad \beta = \frac{m_\mathrm{f}}{m_\mathrm{p} + m_\mathrm{f}} \quad \text{and} \quad \gamma = g\ell^3 \frac{m_\mathrm{p} + m_\mathrm{f}}{EI} \tag{24}$$

According to these definitions, the boundary conditions of (22) must be:

$$\eta(\tau, 0) = \frac{\partial \eta}{\partial \xi}(\tau, 0) = \frac{\partial^2 \eta}{\partial \xi^2}(\tau, 1) = \frac{\partial^3 \eta}{\partial \xi^3}(\tau, 1) = 0 \tag{25}$$

Following [2], consider the case $\gamma = 0$ and assume that $\eta(\tau, \xi) = \mathrm{Re}[Y(\xi)\exp(\mathrm{i}\omega t)]$. Taking a trial solution $Y(\xi) = A\exp(\mathrm{i}\alpha\xi)$, with A constant, and replacing it into equation (22), the admissible values for α must satisfy the following equation:

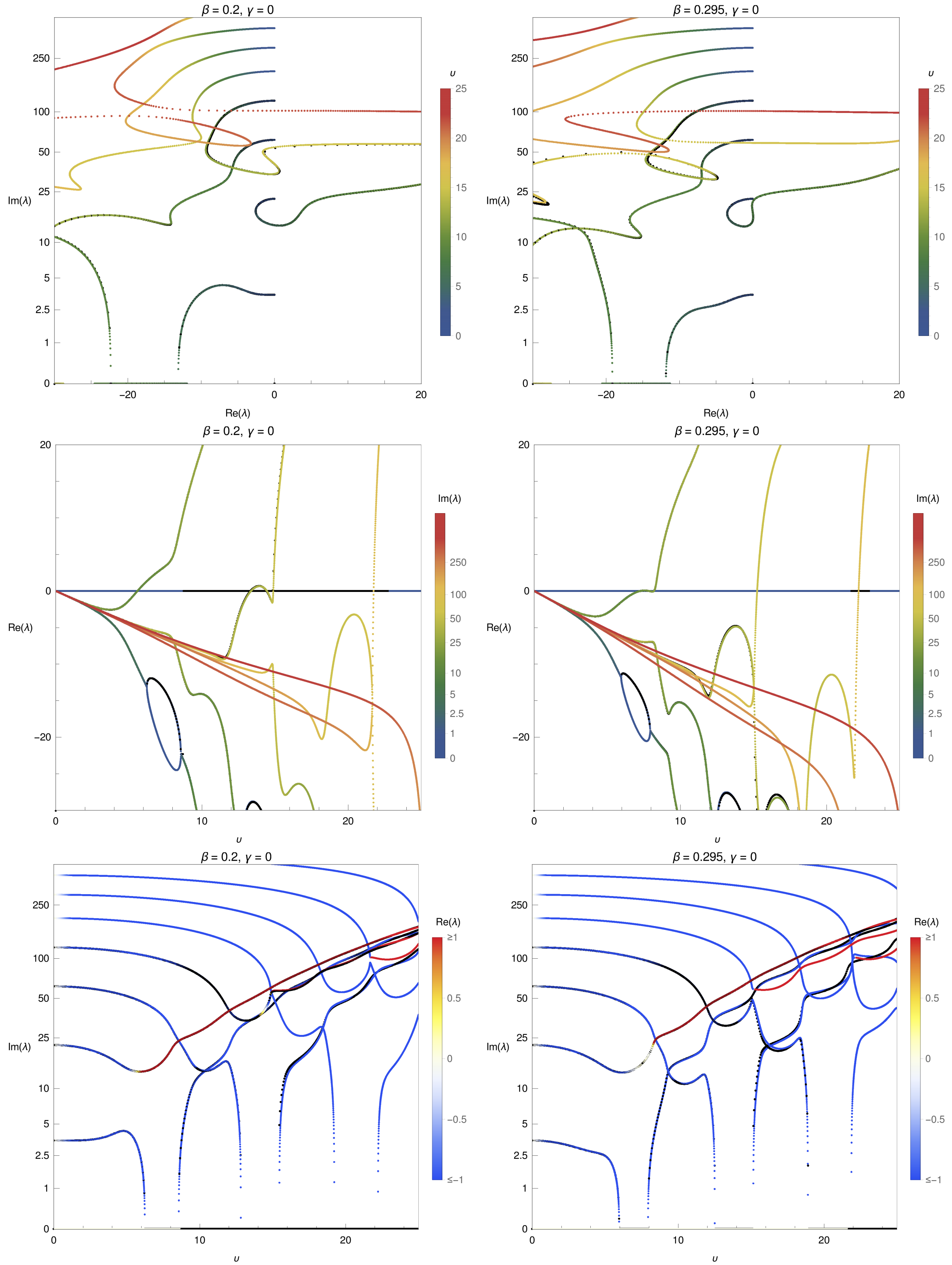
$$\alpha^4 - u^2\alpha^2 - 2\beta^{\frac{1}{2}} u\omega\alpha - \omega^2 = 0 \tag{26}$$

Once (26) is a fourth-order polynomial equation, there must be 4 complex values of α satisfying it, which means that the complete solution of (22),for $\gamma = 0$, is given by:

$$\eta(\tau, \xi) = \mathrm{Re}\left[\sum_{j=1}^{4} A_j \exp(\mathrm{i}\alpha_j \xi)\exp(\mathrm{i}\omega t)\right] \tag{27}$$

Such a solution satisfies the boundary conditions (25) if and only if:

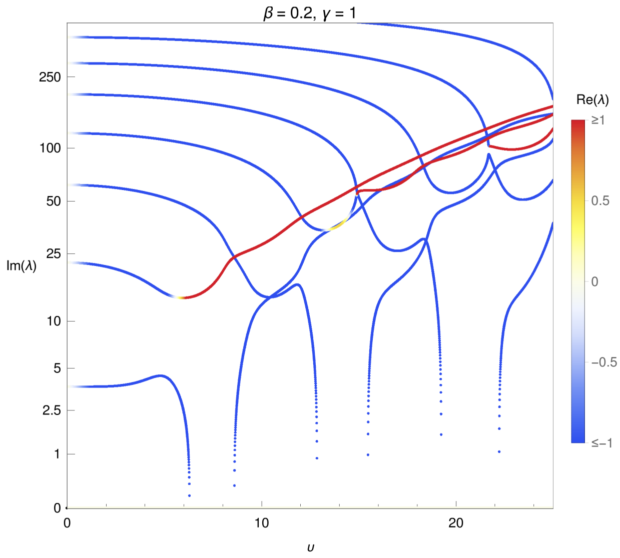
$$\mathit{\Delta} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 \mathrm{e}^{\mathrm{i}\alpha_1} & \alpha_2^2 \mathrm{e}^{\mathrm{i}\alpha_2} & \alpha_3^2 \mathrm{e}^{\mathrm{i}\alpha_3} & \alpha_4^2 \mathrm{e}^{\mathrm{i}\alpha_4} \\ \alpha_1^3 \mathrm{e}^{\mathrm{i}\alpha_1} & \alpha_2^3 \mathrm{e}^{\mathrm{i}\alpha_2} & \alpha_3^3 \mathrm{e}^{\mathrm{i}\alpha_3} & \alpha_4^3 \mathrm{e}^{\mathrm{i}\alpha_4} \end{vmatrix} = 0 \tag{28}$$



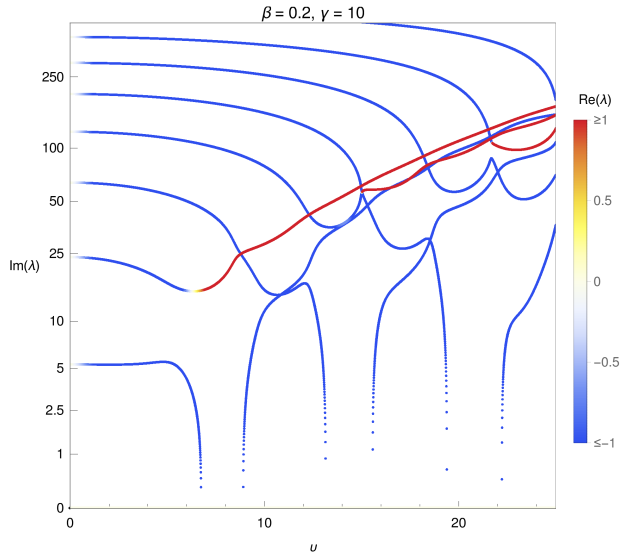
(a) $\beta = 0.2, \gamma = 0$

(b) $\beta = 0.295, \gamma = 0$

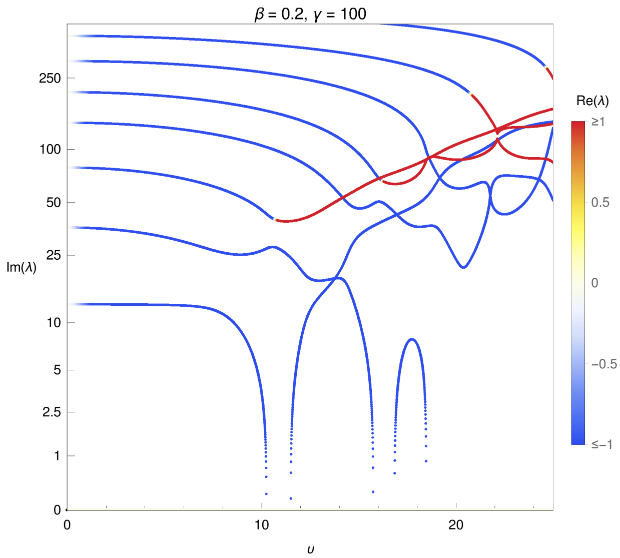
Figure 3: $\mathrm{Im}(\lambda)$ vs. $\mathrm{Re}(\lambda)$, $\mathrm{Re}(\lambda)$ vs. u, and $\mathrm{Im}(\lambda)$ vs. u plots for two reference scenarios: colored dots correspond to the eigenvalues of the linearized reduced order model obtained in this text, black dots to the eigenvalues associated thr first four modes of the linear PDE benchmark model by Gregory and Païdoussis [2].



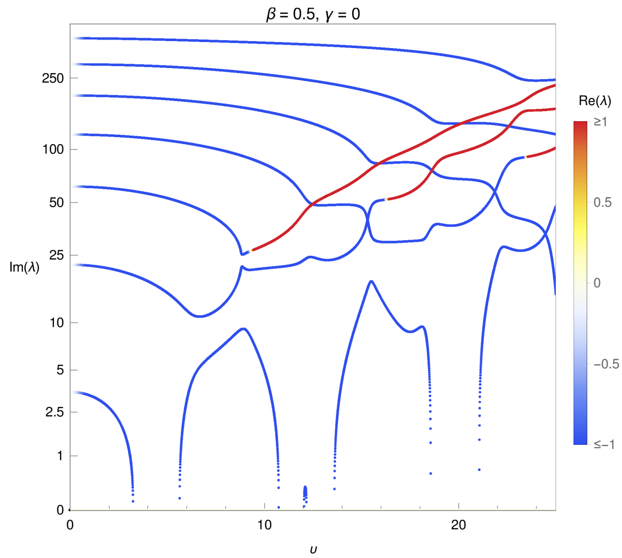
(a) $\beta = 0.2, \gamma = 1$



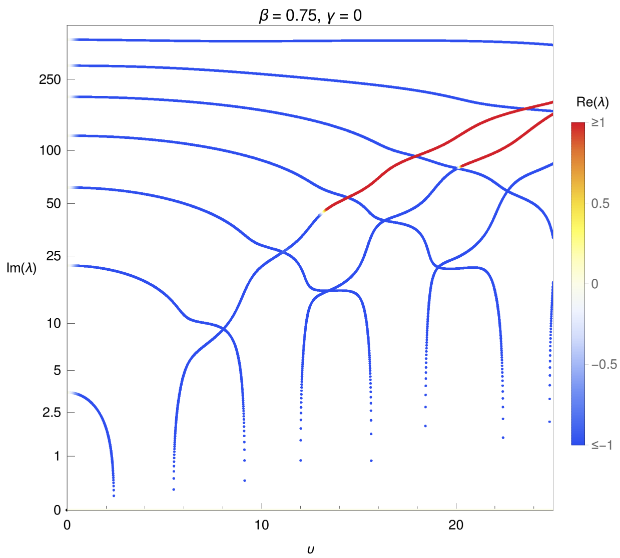
(b) $\beta = 0.2, \gamma = 10$



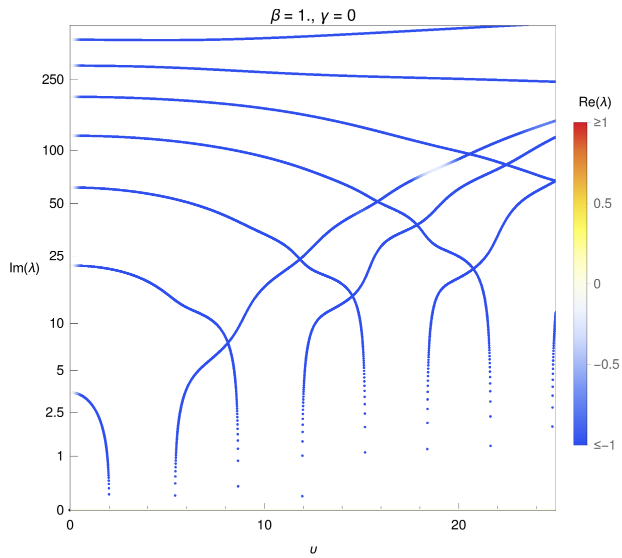
(c) $\beta = 0.2, \gamma = 100$



(d) $\beta = 0.5, \gamma = 0$

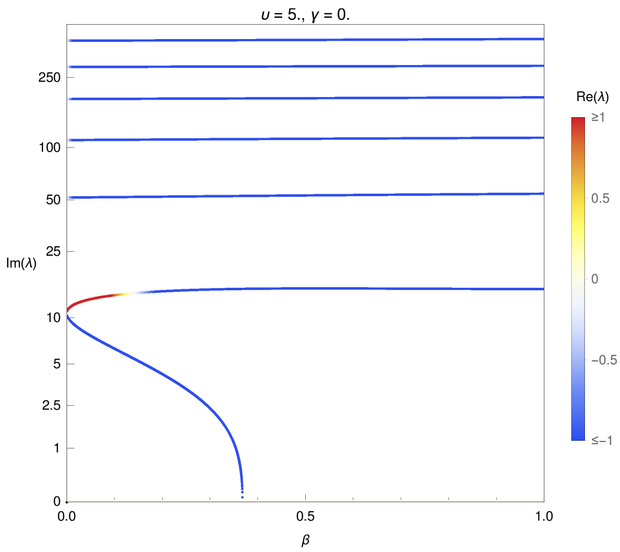


(e) $\beta = 0.75, \gamma = 0$

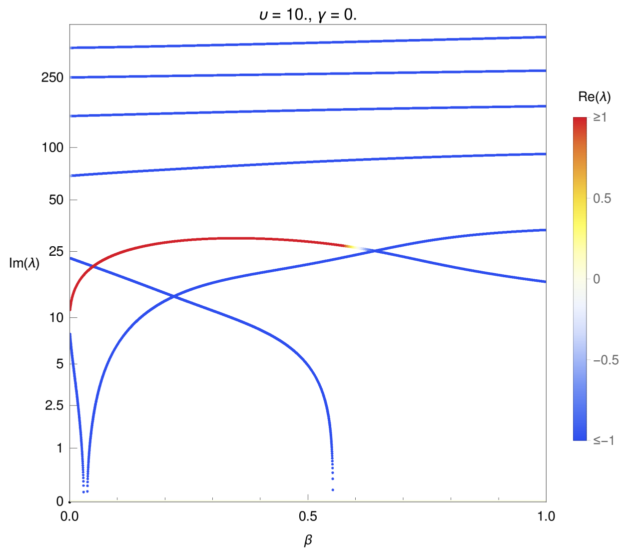


(f) $\beta = 1, \gamma = 0$

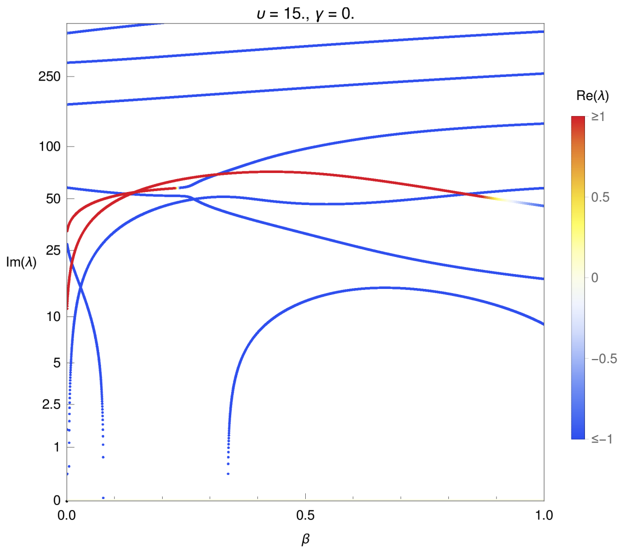
Figure 4: $\text{Im}(\lambda)$ vs. u plots for six scenarios in which the values of β and γ are fixed.



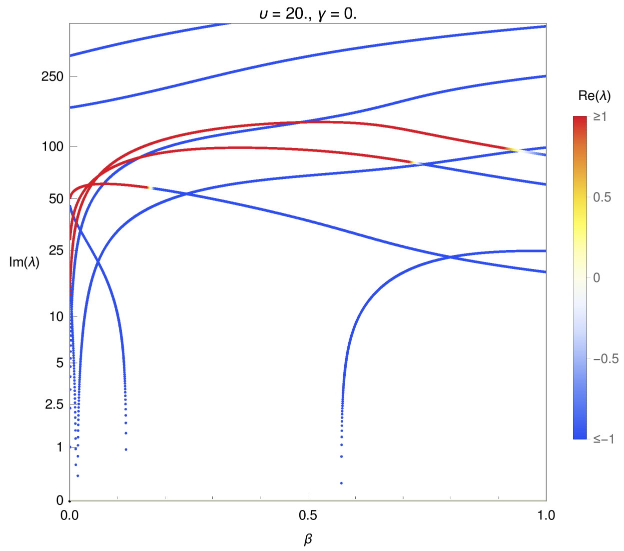
(a) $u = 5, \gamma = 0$



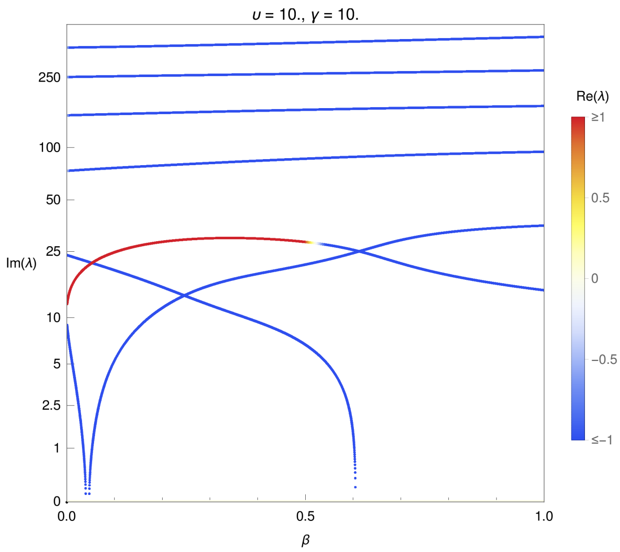
(b) $u = 10, \gamma = 0$



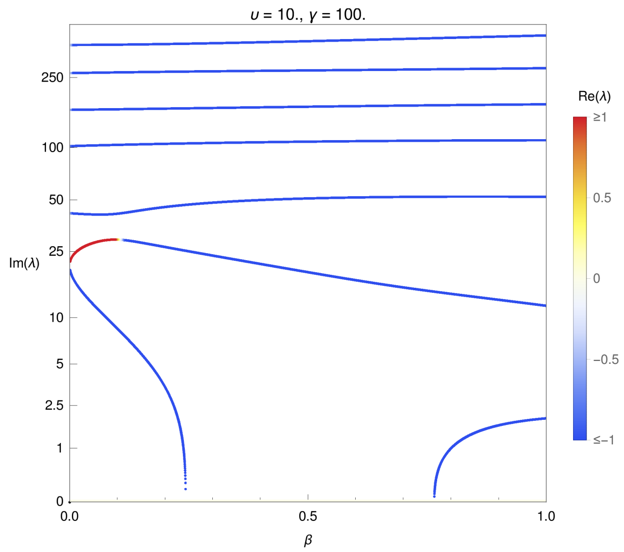
(c) $u = 15, \gamma = 0$



(d) $u = 20, \gamma = 0$



(e) $u = 10, \gamma = 10$



(f) $u = 10, \gamma = 100$

Figure 5: $\mathrm{Im}(\lambda)$ vs. β plots for six scenarios in which the values of u and γ are fixed.

Therefore, in order to describe the eigenvalues of the benchmark model as a function of either β or u (fixing the value of one of these parameters and varying the other within an interval), the following symbolic-numeric algorithm was implemented in Wolfram Mathematica:

(a) Fix a parameter and find the four roots of the polynomial equation (26) as a function of the other one and ω.

(b) Discretize the interval of values for the non-fixed parameter and, for each value among these discrete ones, replace the four roots, now expressed as a function of ω only, into (28). Solve the obtained equation numerically, listing the values of ω in an open ball $|\omega| < \Omega$ for which $\Delta = 0$.

Taking for instance β as a fixed parameter and varying u within a given interval, and considering that, from (27), the eigenvalues λ of the model can be related to the values of ω by the identity $\lambda = \mathrm{i}\omega$, one can use the algorithm described above to obtain plots of $\mathrm{Re}(\lambda) = -\mathrm{Im}(\omega)$ versus u, $\mathrm{Im}(\lambda) = \mathrm{Re}(\omega)$ versus u and even of $\mathrm{Im}(\lambda)$ versus $\mathrm{Re}(\lambda)$ for an interval of values of u. Once the objective of the analysis using the linear model is simply to reproduce the benchmark results presented in [2, 7], this algorithm is only applied to obtain the corresponding plots associated to the first four natural modes for the cases $\beta = 0.2$ and $\beta = 0.295$ (both with $\gamma = 0$). These plots are shown in Figures 3 as black dots superimposed to the colored ones, which are associated to the model developed in this paper.

Analyzing the results shown in Figure 3, it can be noticed that the reduced order model proposed in this text can reproduce with great accuracy the results from the benchmark one, which is given by the PDE (22). The differences among the results become more noticeable for higher values of u and $\mathrm{Im}(\lambda)$. Indeed, if $\gamma = 0$, the projection functions coincide with the actual modes of the cantilevered pipe for $u = 0$, once its linear model reduces to the one of a cantilevered Euler-Bernoulli beam. Thus, for small values of u and $\mathrm{Im}(\lambda)$, the effects of inertial forces are not responsible for significative modifications in the shape of the modes of the system with respect to the projection functions adopted. This, along with the fact that the estimation of the eigenvalues of a distributed parameter system from a reduced order model respects the same stationarity principle underlying the Rayleigh quotient method (i.e. the error in the estimation of the eigenvalues is of quadratic order with respect to the error in the estimation of the modes), explains the good accuracy observed.

In order to better understand the physical meaning of the observed results, it should be noticed that the constraint enforcement performed by Udwadia-Kalaba equation (17) is a projection scheme that effectively creates new amplitude-modulated modal-like functions from the original ones adopted in the Galerkin discretization (2). This modulation is based on the constraints (12), and has a linear correction effect on the influence of gravitational forces in each degree of freedom of the system and a quadratic order correction effect on the influence of inertial and elastic forces, the former being observed in the linearized model and the latter not.

Finally, observing the scenarios shown in Figures 4 and 5, it can be stated that the dynamic response of the system is much more sensible with respect to β than it is to γ, as can be observed in other results in the literature [7]. Variations in β have a huge influence on the individual behavior of each mode and even on which of the modes of the system will be the first to reach a critical speed. It is also worth noting that all the transitions to instability observed in these scenarios characterize Hopf bifurcations.

Conclusions

In this paper, a novel modular approach was applied to to derive a reduced order model of an inextensible cantilevered pipe conveying fluid, a benchmark problem of the area of Fluid-Structure Interaction. Udwadia-Kalaba equation was used for the constraint enforcement algorithm, and using a proper technique, a linearized model was obtained. A numerical analysis was performed in order to characterize the dynamic response of this model by describing the dependency of its eigenvalues with respect to its three nondimensional parameters. The results for two reference scenarios show good agreement with the benchmark values available in the literature. This indicates that, not only should these equations of motion be used for further analyses on the dynamics of a pipe conveying fluid, including nonlinear numerical simulations, but also that the modeling methodology applied might be adequate to the mathematical modeling of other problems in Fluid-Structure Interaction.

Acknowledgments

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