A solution of the general single contact frictionless problem using tools of b-geometry

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Summary. This study provides a new systematic formulation leading to equations of motion for a class of mechanical systems subject to a single frictionless contact constraint. To achieve this goal, some concepts of b-geometry are utilized, setting up a suitable and strong theoretical foundation. A set of ordinary differential equations (ODEs) is first derived for the case of minimal coordinates, which can be viewed as the projection of the equation of motion along the normal direction to the boundary of the configuration space to all other directions. Then, a comparison with the equations obtained for systems subject to bilateral constraints is performed. Finally, the basic new theoretical concepts are illustrated by two selected examples.

Introduction - Objectives

The dynamics of constrained mechanical systems is a classical and beautiful subject of Analytical Mechanics, which has received the continuous attention of the engineering research community, over a long period of time (e.g., [1-6]). Despite all the efforts, there still remains a plethora of challenging theoretical aspects on the subject that remain open. Consequently, new studies in this area are of large practical significance, since their successful outcome can cause an impact in formulating and solving complicated engineering problems. Traditionally, this subject has been divided in two major areas. In the first, the constraints affecting the motion of a system can be expressed by equalities and give rise to forces which can be compressive or tensile. For this reason, such constraints are called bilateral. In the second area, the constraints involve contact with or without friction and generate forces which can act in one direction only, in order to avoid interpenetration. Such constraints are described by inequalities and are known as unilateral.

Currently, there exists a vast literature on the subject of unilateral constraints. Based on the type of analysis performed, the available approaches can roughly be divided in two groups. In the first group, the research is conducted by applying classical analytical tools [7-11], while the work in the second group is more heavily based on ideas of non-smooth mechanics involving convex analysis [12-16]. In brief, these methodologies apply the classical laws of motion up to a time where contact is established between the interacting bodies. Then, based on the short duration of the impact phase following the contact, it is assumed that the position of the system components does not change appreciably, while the forces developed at the contact are excessively large, causing a significant change in the velocities of the impacting bodies. This brings the need to introduce an impact law, predicting the sudden change in the velocities, considered as a shock. Typically, such a change in the velocities is predicted by using Newton's or Poisson's restitution coefficients or other similar concepts [8, 11].

In the great majority of the previous studies on the subject, the approach leading to prediction of the post-impact velocity is algebraic in nature. However, in some cases, the time interval where the contact event takes place is considered to be short but finite and the contact process is modeled by an approximate set of ODEs, with the normal impulse playing the role of a time-like parameter [7, 9-11]. This is known as a Darboux-Keller approach and is currently confined to contact between two particles or rigid bodies [14]. The present work can be considered as a first step towards extending and generalizing this type of analysis to systems possessing general dynamic properties, by adopting the general framework of Analytical Dynamics. Specifically, the main objective of this work is to present a new formulation for an important and practically significant class of problems, involving unilateral constraints. Building upon the strong relation between mechanics and differential geometry [17, 18], the methodology developed is based on some recent work of the authors on mechanical systems subject to bilateral constraints [19, 20], together with some new geometric concepts. Namely, some powerful tools of b-geometry are utilized for setting up a suitable framework in studying mechanical systems subject to unilateral constraints [17, 21, 22]. In particular, the present study focuses on the investigation of a class of systems involving a single frictionless contact. In this way, several new mathematical ideas are introduced and explained in a problem involving a sufficiently high but not excessive level of mechanical complexity. Moreover, the analysis is performed in a way providing a solid foundation for a subsequent extension and application to more complex mechanical problems, involving friction or even multiple contact events [12-15].

The present work takes a different perspective and route than previous studies on the subject. The main difference is that Newton's second law of motion is applied even during the relatively short time interval where a contact event occurs between two of the interacting bodies. In this way, there is no need to define extra concepts, like the restitution coefficient. However, the most fundamental aspect of the present work is that the analysis performed remains fully smooth. Namely, the generalized velocities and the corresponding momenta remain smooth and bounded during the entire contact phase. This is achieved by using some remarkable tools of b-geometry, referring to the theory of manifolds with boundary and the introduction of an appropriate vector bundle over the constrained configuration manifold, consisting of smooth velocity fields only [21, 22]. Consequently, the new approach provides a strong theoretical basis for a systematic formulation leading to the equations of motion for mechanical systems involving a single frictionless contact. A set of ODEs is first derived for the case of a minimal set of coordinates. These equations are found to describe the action in directions normal and tangent to the boundary, in a fundamentally different manner. Then, these equations are compared to those obtained for systems subject to bilateral constraints. Finally, the basic new

theoretical concepts are illustrated and reinforced further by a selected set of mechanical examples. Apart from a more accurate and realistic modeling of the impact phase, the results of the present analysis are expected to lead to the development of more efficient and robust numerical schemes for mechanical systems involving contact.
The organization of this paper is as follows. First, some basic concepts are summarized in Section 2, illustrating the relation between the theory on manifolds with boundary and mechanical systems with a unilateral constraint. The essential geometric properties of such manifolds are also presented. This provides a strong theoretical basis for a systematic formulation leading to the equations of motion, included in Section 3. A comparison to systems subject to bilateral constraints is also presented. Finally, two characteristic examples are examined in Section 4.

Mechanical systems involving contact and geometric properties of manifolds with boundary

This study focuses on the dynamics of mechanical systems involving collision between their components. A single contact event is possible to occur at a time, which is frictionless. The motion is described by a set of generalized coordinates, $q=(q^1,\dots,q^n)$, corresponding to a point p, moving as a function of time t on the n-dimensional configuration manifold M of the unconstrained system [3]. The correspondence between point p and its coordinates q is established by a mapping from a neighborhood of p to the Euclidean space $\mathbb{R}^n$, with form

$$q=\varphi(p)\,, \tag{1}$$

known as a coordinate map [17]. Here, the set of coordinates selected is assumed to be minimal.
The possibility of contact between the components of the system is described by an inequality condition

$$\rho(p)\geq 0\,, \tag{2}$$

representing a unilateral constraint and defining a hypersurface, which is embedded in the original manifold and restricts the motion of the figurative particle p in one side of this hypersurface only [21]. This generates a manifold X, which possesses a boundary ∂X defined by the equality in condition (2) and an interior $X^o = X\setminus\partial X$, represented by the disjoint union $X = X^o \coprod \partial X$. Then, the motion of the system is represented by a curve on manifold X. Also, a tangent vector to the curve at a point p belongs to an n-dimensional vector space T_pX, the tangent space at p. Therefore, if $\mathfrak{B}_e=\{\underline{e}_1 \quad \dots \quad \underline{e}_n\}$ is a basis of T_pX and after employing the following summation convention

$$u^I\underline{e}_I=\sum\nolimits_{I=1}^{n}u^I\underline{e}_I \quad \text{and} \quad u^i\underline{e}_i=\sum\nolimits_{i=2}^{n}u^i\underline{e}_i\,,$$

any of its elements can be put in the form

$$\underline{u}=u^I\underline{e}_I=u^1\underline{e}_1+u^i\underline{e}_i\,, \tag{3}$$

with $I=1,\dots,n$ and $i=2,\dots,n$. Then, the tangent vector bundle over X is defined by

$$TX=\coprod_{p\in X}T_pX\,. \tag{4}$$

If $V(X)$ is the space of all smooth vector fields on X, its elements are not well-behaved on X since integration to obtain the corresponding flows is not closed on X. Fortunately, there exists a remarkable remedy, based on the theory of manifolds with boundary [21]. According to this theory, one has to consider a smaller space of smooth vector fields on X, which are tangent to the boundary. Namely, these vector fields are elements of a new space, defined by

$$V_b(X)\equiv\{V\in V(X):\ V \text{ is tangent to } \partial X\}\,. \tag{5}$$

This means that if the coordinates $x=(x^1,\dots,x^n)$ are introduced at a point p of the boundary ∂X, so that they all vanish at p, i.e., $\varphi(p)=0$, then any element of a vector field belonging to $V_b(X)$ can be put in the form

$${}^b\underline{v}=\alpha x^1\underline{g}_1+a^i\underline{g}_i\,, \tag{6}$$

over a holonomic basis $\mathfrak{B}_g=\{\underline{g}_1,\dots,\underline{g}_n\}$ near the boundary, with $\rho=x^1\geq 0$. Therefore, the elements $x^1\underline{g}_1$ and $\underline{g}_i$ form a spanning set for $V_b(X)$ near the boundary. Then, it can be shown that there exists a vector bundle ${}^bTX$ over X, the b-tangent bundle, which does not suffer from the drawbacks of TX [13]. This and its dual bundle will be used in the following section as a domain and image space for evaluating the derivatives in the law of motion.
Next, the emphasis is placed on determining the essential geometric properties of the configuration manifold with boundary X. In analogy to classical Riemannian geometry, a b-metric is expressed in the following form

$${}^bg=\bar{g}_{IJ}\,\underset{\sim}{\bar{g}}^I\otimes\underset{\sim}{\bar{g}}^J$$

with respect to any basis ${}^b\mathfrak{B}^*_{\bar g}$ of the coordinate system originating from the local coordinates x at the boundary, with $\bar g_{IJ} = \langle \underline{g}_I, \underline{g}_J \rangle$. First, at points away from the boundary ∂X,

$$\bar g_{IJ} = g_{IJ}\,, \tag{7}$$

where g_{IJ} are the components of the metric on manifold M. On the other hand, at points near the boundary ∂X

$$\bar g_{IJ} = g_{IJ} + \hat g_{IJ}\,, \tag{8}$$

in the coordinate system examined, with

$$\hat g_{11} = \widehat{g}_{11} \big/ (x^1)^2 \quad \text{and} \quad \hat g_{1j} = \hat g_{ij} = 0\,. \tag{9}$$

Moreover, employing the classical metric compatibility condition [23], it can be shown that

$$\widehat{g}_{11} = b^2 g_{11}\,. \tag{10}$$

If the transformation between the original q-coordinate system and the x-coordinate system is expressed in the form

$$\dot x^I = A^I_{I'} \dot q^{I'}\,, \quad (I, I' = 1, \ldots, n)\,, \tag{11}$$

then, the metric components in the x-coordinate system are determined by

$$g_{IJ} = B^{I'}_I B^{J'}_J g_{I'J'}\,, \tag{12}$$

where $g_{I'J'}$ are the components of the metric with respect to the q-coordinate system. Also, $B = [B^{I'}_I]$ is the matrix inverse to $A = [A^I_{I'}]$. Therefore, their components are related by

$$A^I_{K'} B^{K'}_J = \delta^I_J \quad \text{and} \quad B^{I'}_K A^K_{J'} = \delta^{I'}_{J'}\,. \tag{13}$$

Then, the single term $\hat g_{11}$ is spread all over the metric matrix through the transformation

$$\hat g_{I'J'} = A^1_{I'} A^1_{J'} \hat g_{11} \tag{14}$$

and the components of the exact b-metric $\bar g_{I'J'}$ with respect to the original basis of $T^*_p X$ can be put in the form

$$\bar g_{I'J'} = g_{I'J'} + \hat g_{I'J'}\,. \tag{15}$$

The elements of the first line of matrix $A = [A^I_{I'}]$, needed for determining the $\hat g_{I'J'}$ by Eq. (14), are evaluated by using the fact that the gradient vector $\nabla\rho$ should be normal to the boundary hypersurface defined by $\rho = 0$. Then,

$$A^1_{I'} = \langle \nabla\rho, \underline{e}_{I'} \rangle\,. \tag{16}$$

In analogy to the material presented for the b-metric, Eq. (8), the b-affinities are also decomposed in the form

$$\bar\Lambda^K_{IJ} = \Lambda^K_{IJ} + \mathrm{A}^K_{IJ}\,, \tag{17}$$

with respect to any basis ${}^b\mathfrak{B}_{\bar g}$ of ${}^bT_pX$. The terms Λ^K_{IJ} are the affinities of the ordinary vector bundle TX, defined over the manifold M, with respect to a basis $\mathfrak{B}_g$ of T_pX, while A^K_{IJ} are terms signaling the presence of the manifold boundary. The latter terms are negligible away from ∂X, so that

$$\bar\Lambda^K_{IJ} = \Lambda^K_{IJ} \quad \text{over } X^o\,, \tag{18}$$

meaning that the b-affinities coincide with the ordinary affinities away from the boundary. To see what happens near the boundary ∂X, a local x-coordinate system is selected for convenience, once again. Then, based on the velocity transformation expressed by Eq. (11), the ordinary affinities are evaluated in the basis ${}^b\mathfrak{B}_g$ by

$$\Lambda^K_{IJ} = B^{I'}_I B^{J'}_J A^K_{K'} \Lambda^{K'}_{I'J'} + A^K_{K'} B^{K'}_{J,I}\,, \tag{19}$$

where $\Lambda^{K'}_{I'J'}$ are the affinities with respect to the original basis $\mathfrak{B}_e$ of T_pX. In addition, starting from Eq. (12) and employing Eq. (13), it can easily be shown that

$$B^{I'}_1 = g^{I'J'} A^1_{J'} g_{11} \quad \text{and} \quad B^{I'}_{1,1} = B^{I'}_{1,I'} B^{I'}_1\,, \tag{20}$$

since $g^{I'J'} g_{J'K'} = \delta^{I'}_{K'}$. Determination of the correction terms A^K_{IJ} in Eq. (17) is more involved. First, it is straightforward to show that these boundary terms are components of a tensor, by noting that the b-affinities $\bar\Lambda^K_{IJ}$ transform in an exactly similar way as the ordinary affinities Λ^K_{IJ}, governed by Eq. (19). Then, based on Eq. (11), it turns out that

$$\mathrm{A}^K_{IJ} = B^{I'}_I B^{J'}_J A^K_{K'} \mathrm{A}^{K'}_{I'J'}\,, \quad \text{with} \quad \mathrm{A}^{K'}_{I'J'} = \bar\Lambda^{K'}_{I'J'} - \Lambda^{K'}_{I'J'}\,. \tag{21}$$

Moreover, by employing Eq. (13), the last relation furnishes these terms in the original q-coordinate system, given A_{IJ}^{K} in a local x-coordinate system. These steps complete the process of determining the components of the exact b-affinities $\bar{\Lambda}_{I'J'}^{K'}$ with respect to the original basis of $T_p X$, through Eqs (11) and (13).

Equations of motion for the single contact frictionless problem

The natural trajectory of a system on its configuration manifold is determined through application of Newton's law of motion [18,19]. On a manifold without boundary, this law is expressed in the form

$$\nabla_{\underline{v}} \underset{\sim}{p}^{*} = \underset{\sim}{f}^{*}, \tag{22}$$

where ∇ represents an affine connection on the manifold, so that the term in the left hand side represents the covariant differential of the generalized momenta $\underset{\sim}{p}^{*}$ along a path on the manifold with tangent vector $\underline{v}$. As usual,

$$\nabla_{\underline{v}} \underset{\sim}{p}^{*} = (\dot{p}_I - \Lambda_{JI}^{L} \, p_L v^J) \underset{\sim}{e}^{I} \quad \text{with} \quad p_I = g_{IJ} v^J , \tag{23}$$

while $\underset{\sim}{f}^{*}$ stands for the applied forces [6]. Next, this law is applied on the constrained manifold X in the form

$${}^{b}\nabla_{\underline{\bar{v}}} \underset{\sim}{\bar{p}}^{*} = \underset{\sim}{\bar{f}}^{*}, \tag{24}$$

since the appropriate vector and covector quantities live in ${}^{b}T_p X$ and ${}^{b}T_p^{*} X$, respectively. Within a layer close to the boundary ∂X of X, with a small width b, both the metric components and the affinities are affected in a significant manner by the presence of the boundary, leading to the necessity of applying a boundary layer analysis [24].

The laws of motion expressed by Eqs (22) and (24) are identical in the interior X^{o} of manifold X. Inside the boundary layer, the coordinate x^1 is of the order of the layer width b, which is much smaller than any characteristic length of the problem. Also, the action in the direction normal to the boundary is not coupled with the motion in the tangent plane to the boundary hypersurface. Likewise, no impulse occurs in the tangential directions and no coupling exists with the motion in the normal direction. In addition, the forcing terms f_I (with $I = 1,\ldots,n$) are selected to be of small order, while a forcing term $\hat{f}_1$ is introduced, so that it is negligible away from the boundary and

$$\hat{f}_1(x^1, \dot{x}^1) = \left[\frac{k_1}{x^1} - \frac{c_1 \dot{x}^1}{(x^1)^2}\right] \hat{s}(x^1; a, b) \tag{25}$$

at the boundary. The function $\hat{s}(x^1; a, b)$ guarantees a smooth transition of the boundary force from the inner to the outer region of the boundary layer. A possible definition of such a function is as follows

$$\hat{s}(x; a, b) = 1 - s(x; a, b),$$

with

$$s(x; a, b) = s(\tfrac{x-a}{b-a}), \quad s(x) = \frac{f(x)}{f(x) + f(1-x)} \quad \text{and} \quad f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

Then, keeping only the dominant terms within the boundary layer, it turns out that

$$\bar{h}_1 = \dot{\hat{p}}_1 - \bar{\Lambda}_{11}^{1} \hat{p}_1 v^1 - \hat{f}_1 = 0 \quad \text{and} \quad \bar{h}_i = \dot{p}_i - \Lambda^{\ell}_{\;ji} \, p_\ell v^j - f_i = 0 , \tag{26}$$

along the normal and tangential directions of the boundary, respectively. Finally, by employing Eq. (11), the equations of motion can be transformed to component form with respect to the original q-coordinate system, with form

$$\dot{\bar{p}}_{I'} - \bar{\Lambda}_{J'I'}^{L'} \bar{p}_{L'} v^{J'} - \bar{f}_{I'} = 0, \quad \text{for} \quad I' = 1,\ldots,n . \tag{27}$$

These constitute a coupled set of n ODEs. In order to see in a better way what happens near the boundary, by a simple manipulation of the last equation and omitting the smaller order terms, the last equation yields eventually

$$A_{I'}^{1} (\dot{\hat{p}}_1 - \bar{\Lambda}_{11}^{1} \hat{p}_1 v^1 - \hat{f}_1) = 0 . \tag{28}$$

Therefore, each of these terms represents a projection of the equation of motion along the normal direction to the boundary to all directions of the q-coordinate system.

In contrast to unilateral constraints, bilateral constraints are expressed as equalities, with general form

$$\dot{\psi}^{R}(q, \dot{q}) \equiv a_{I'}^{R}(q) \dot{q}^{I'} = 0 , \quad (I' = 1,\ldots,n \quad \text{and} \quad R = 1,\ldots,k), \tag{29}$$

in the original q-coordinate system. In some cases, these equations can be integrated and put in the form

$$\phi^R(q) = 0\,. \tag{30}$$

According to results presented in a recent study [19], the equations of motion of such systems can be put in the form

$$\dot{p}_{I'} - \Lambda^{L'}_{J'I'} p_{L'} v^{J'} - f_{I'} = \sum_{R=1}^{k} a_i^R [(\overline{m}_{RR}\dot{\lambda}^R)^{\cdot} + \overline{c}_{RR}\dot{\lambda}^R + \overline{k}_{RR}\lambda^R - \overline{f}_R]\,, \tag{31}$$

where λ^R represents the coordinate of a single dimensional manifold M_R. In the last relation and in the sequel, the convention on repeated indices does not apply to R. Moreover, the remaining parameters are determined by

$$\overline{m}_{RR} = c_R^{I'} \overline{g}_{I'J'} c_R^{J'}\,,\;\; \overline{c}_{RR} = -c_R^{I'} f_{I',J'}(q,\dot{q},t) c_R^{J'}\,,\;\; \overline{k}_{RR} = -c_R^{I'} f_{I',J'}(q,\dot{q},t) c_R^{J'}\,,\;\; \overline{f}_R = c_R^{I'} \overline{f}_{I'}(q,\dot{q},t)\,, \tag{32}$$

with the components of the special n-vector $\underline{c}_R$ selected so that they satisfy the condition

$$\underset{\sim}{q}^R(\underline{c}_R) = 1 \quad \Rightarrow \quad a_{I'}^R c_R^{I'} = 1\,. \tag{33}$$

In general, Eq. (31) leads to a set of n second order coupled ODEs in the $n+k$ unknowns $q^{I'}$ and λ^R. The additional information needed for a complete mathematical formulation is obtained by the k equations of the constraints [19]

$$(\overline{m}_{RR}\dot{\phi}^R)^{\cdot} + \overline{c}_{RR}\dot{\phi}^R + \overline{k}_{RR}\phi^R = 0 \quad \text{or} \quad (\overline{m}_{RR}\dot{\psi}^R)^{\cdot} + \overline{c}_{RR}\dot{\psi}^R = 0\,. \tag{34}$$

These results demonstrate that the equations of motion of a system with bilateral constraints constitute a set of ODEs on the original configuration manifold, but with different form than Eq. (26) or (27). Also, the effect of the unilateral constraint is realized as a modification of the geometric properties of the manifold and not as a restriction on the degrees of freedom. Therefore, in contrast to the action of bilateral constraints, a unilateral constraint does not reduce the dimension of the original configuration manifold but only its extent.

Examples

Two examples are presented in this section. In the first one, the dynamics of Eq. (26) is investigated thoroughly, while the second example serves as a vehicle for illustrating the sequence of the basic steps leading to the equations of motion.

Plane collision of a particle with a rigid wall

The mechanical system considered first consists of a single particle with a mass m, moving in a plane and hitting a rigid wall. Therefore, the particle position can be determined by two Cartesian coordinates, say q^1 and q^2, while the original configuration space is $M = \mathbb{R}^2$. Here, the x-coordinate system is chosen to coincide with the q-coordinate system. Also, the corresponding metric matrix and the affinities away from the boundary are

$$G = [g_{IJ}] = mI_2 \quad \text{and} \quad \Lambda^K_{IJ} = 0, \quad (I,J,K = 1,2),$$

respectively, where I_2 is the 2×2 identity matrix, while the wall is located at $q^1 = 0$. Therefore, the boundary defining function is the following

$$\rho(q) = q^1\,.$$

Moreover, the particle hits the wall with negligible friction and an initial velocity $\underline{v}_0 = (-v^- \quad v_T)^T$.

First, the action along the normal direction to the boundary is determined by solving Eq. (26). For this, the parameters k_1 and c_1 in Eq. (25) for the boundary induced force are scaled in the form

$$k_1 = kb^2 \quad \text{and} \quad c_1 = cb^2/v^-\,.$$

Then, Eq. (26) can be put in the form

$$\Big[\frac{g_{11}}{(x^1)^2} v^1\Big]^{\cdot} - \frac{g_{11}}{(x^1)^3}(v^1)^2 - \frac{k}{x^1} + \frac{cv^1}{v^-(x^1)^2} = 0\,, \tag{35}$$

with $g_{11} = m$ and $v^1 = \dot{x}^1$. Using the initial conditions $x^1(0) = a$ and $v^1(0) = -v^-$, the solution of this second order nonlinear ODE in x^1 can be expressed in the form

$$x^1(t) = ae^{\beta(t)} \quad \text{and} \quad v^1(t) = v^-[(e^{-ct/(v^- g_{11})} - 1)ak/c - 1]e^{\beta(t) + ct/(v^- g_{11})} \equiv v(t;k,c)\,, \tag{36}$$

with

$$\beta(t) = (v^-)^2[(k + c/a)g_{11}(e^{ct/(v^- g_{11})} - 1) - ckt/v^-]/c^2\,.$$

The analytical solution is convenient in demonstrating the effect of the spring-like parameter k and the damper-like parameter c during the contact phase. Some characteristic results along these lines are presented next. All these results were obtained by assuming a unit value as a reference value for the parameters b, m, k and c, corresponding to a suitable normalization.

First, in Fig. 1a are presented results for the history of the displacement component along the normal direction to the boundary, for $c = 0$ and several values of the spring-like parameter k. The results indicate that an increase in the value of k causes a reduction in the penetration depth to the boundary layer. In all cases, the diagrams are symmetric with respect to the line $t = t_c$, corresponding to the time where the velocity component v^1 normal to the boundary becomes zero. In classical formulations, the time t_c signals the end of the compression phase. Also, due to the absence of energy dissipation effects, the compression and expansion phases have equal duration. Finally, the histories of the corresponding velocities are shown in Fig. 1b. Clearly, the effect of k is more pronounced on the form of the velocity histories for relatively small values of this parameter.

Likewise, in Fig. 1c are presented results for the displacement along x^1, for $k = 1$ and several values of the damper-like parameter c. Here, an increase in the value of c causes a reduction in the penetration depth, once again. However, it also causes a break in the symmetry of the diagrams with respect to the line $t = t_c$. This becomes more evident as the dissipation effects, quantified by the coefficient c, become stronger. In fact, for excessive values of c, the normal velocity at the end of the contact phase tends to zero, gradually, approaching conditions of plastic contact. These effects are illustrated in a better way in Fig. 1d, where the corresponding velocity histories are shown.

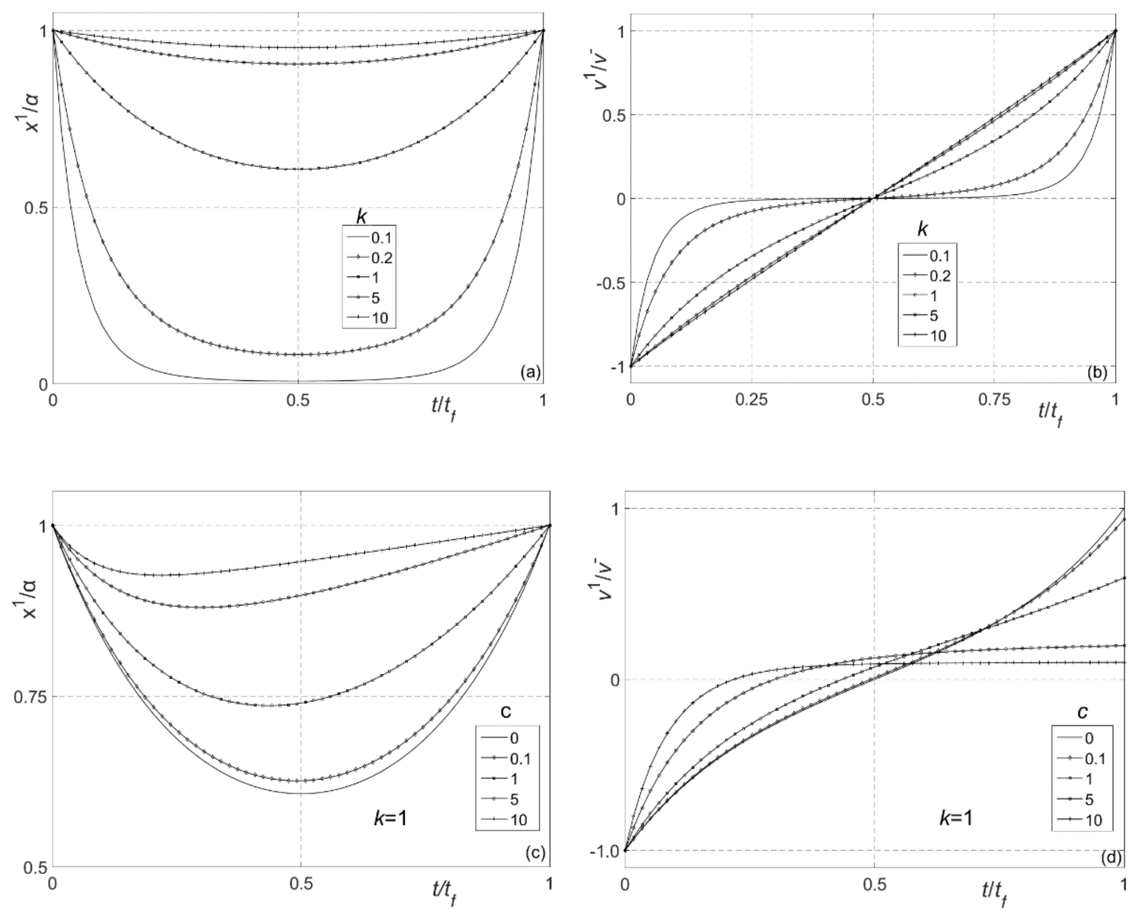

Fig. 1. Effect of the boundary forcing parameter k on the figurative particle (a) displacement and (b) velocity normal to the wall, for $c = 0$. Effect of parameter c on the (c) displacement and (d) velocity normal to the wall, for $k = 1$.

The results presented in Fig. 2 reveal more on the effect of the boundary layer forcing parameters k and c. First, in Fig. 2a is illustrated their effect on the duration of the contact phase, t_f. Specifically, the results presented indicate that an increase in the value of either k or c causes a reduction in the duration of the contact phase. In addition, based on the results of Fig. 1d, it becomes clear that an increase in the value of c causes a decrease in the amplitude of the velocity of the particle when it exits the boundary layer. This indicates dissipation of the kinetic energy. In order to better quantify this effect and relate it to previous studies, in Fig. 2b are presented results obtained by evaluating the corresponding classical kinematic and dynamic restitution coefficients

$$e_N = v^+ / v^- \quad \text{and} \quad e_P = p_2 / p_1 , \tag{37}$$

defined by Newton and Poisson, respectively [11, 14]. In the last expressions, v^+ represents the normal component of the velocity of the particle as it leaves the boundary layer, at $t = t_f$. Moreover,

$$p_1 = \int_0^{t_c} f(\tau) d\tau \quad \text{and} \quad p_2 = \int_{t_c}^{t_f} f(\tau) d\tau , \tag{38}$$

where $f(t)$ is the force exerted on the particle from the wall. In fact, setting $t = t_f$ in Eq. (36) provides an analytical expression for the restitution coefficient e_N. The results obtained indicate that the values of e_N and e_P are virtually indistinguishable for the same value of k and c, which is consistent with previous studies on systems with no friction [14]. Also, the restitution coefficients have value equal to one for $c = 0$ and all values of k, corresponding to cases of an elastic impact, while they take a value less than one for $c > 0$. In addition, an increase in the value of k leads to an increase in the value of the restitution coefficient, which is opposite to the effect of the dissipation parameter c.

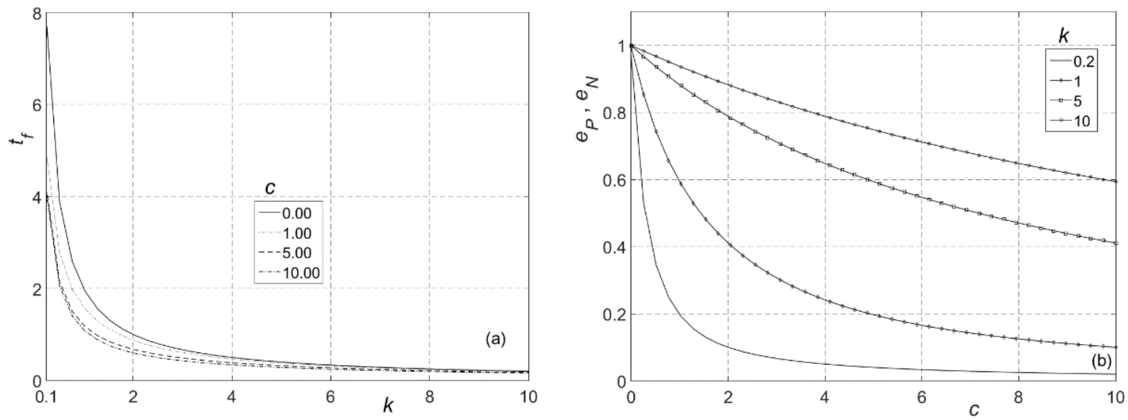

Fig. 2. Effect of parameters k and c on (a) duration of contact phase and (b) restitution coefficients.

Finally, the results depicted in Fig. 3 illustrate the effect of the ratio a/b, selected in the cut off function appearing in Eq. (25). First, in Fig. 3a are shown results for the history of the displacement component along the normal direction to the boundary, for the nominal values $k = 1$ and $c = 1$ and several values of the ratio a/b. Likewise, in Fig. 3b are shown results for the corresponding velocity component. The results demonstrate that a rapid convergence of the numerical results to the analytical solution is observed in the limit $a/b \to 1$.

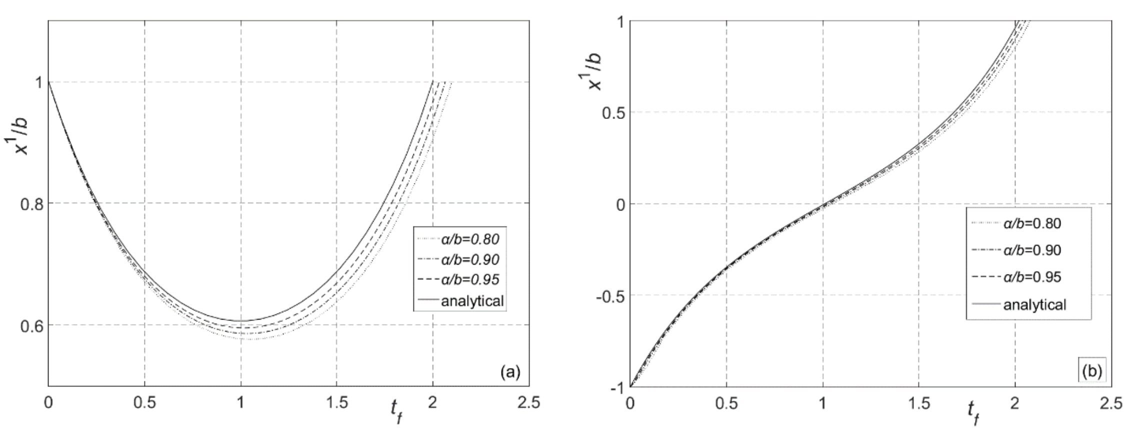

Fig. 3. Effect of ratio a/b on the particle (a) displacement and (b) velocity normal to the wall.

Spatial collision of a rigid body with a rigid wall

In the second example, a rigid body hitting a rigid wall in the absence of friction is examined. In general, the configuration space of a free rigid body can be represented by a six dimensional product configuration space $M \equiv \mathbb{R}^3 \times M(3)$ [25]. This means that the position and orientation of the body with respect to an inertial reference frame $\mathbb{F}$, at any time t, can be represented by a point on this manifold, with generalized coordinates $\underline{q}(t)$ split in two parts, $\underline{q}_C$ and $\underline{q}_R$. The part $\underline{q}_C$ includes the coordinates of the center of mass C of the body with respect to $\mathbb{F}$, while $\underline{q}_R$ is associated to the rotational motion of the body. Then, the velocity vector is written as

$$\underline{v}(t) = (\underline{v}_C^T \quad \underline{v}_R^T)^T ,$$

with $\underline{v}_C = \dot{\underline{q}}_C(t)$. For the rotational component of the rigid body velocity, it is convenient to introduce a set of quasi-coordinates $\underline{\vartheta}$ in place of $\underline{q}_R$ so that

$$\underline{v}_R = \dot{\underline{\vartheta}} \equiv (\Omega^1 \quad \Omega^2 \quad \Omega^3)^T \quad \text{and} \quad \dot{\underline{\vartheta}} = T(\underline{q}_R)\dot{\underline{q}}_R , \tag{39}$$

where $T(\underline{q}_R)$ is the tangent operator at $\underline{q}_R$ [26]. Based on the expression for the kinetic energy of the body, the metric on space M can then be selected in the following block diagonal form

$$G_q = [g_{I'J'}] = \begin{bmatrix} mI_3 & 0 \\ 0 & J_C \end{bmatrix}, \tag{40}$$

where m is the mass and J_C is the mass moment of inertia matrix of the body with respect to a frame fixed in the body, with origin at its center of mass and an orthonormal basis { $\underline{e}_1 \quad \underline{e}_2 \quad \underline{e}_3$ }. Moreover, the only non-zero affinities $\Lambda^{K'}_{I'J'}$ are related to the rotational part of the motion and are selected to take the following constant values

$$\Lambda^{4'}_{5'6'} = -\Lambda^{4'}_{6'5'} = \Lambda^{5'}_{6'4'} = -\Lambda^{5'}_{4'6'} = \Lambda^{6'}_{3'5'} = -\Lambda^{6'}_{5'3'} = 1 . \tag{41}$$

Next, consider a point P on the body, which at some instance comes in contact with a plane rigid wall Π, as shown in Fig. 4. This plane is defined by

$$s(\underline{x}) = \sum_{i=1}^{3} s_i x^i = 0 , \tag{42}$$

where $\underline{x}$ is the position vector of a point on this plane with respect to $\mathbb{F}$. In addition, the position vector of point P with respect to frame $\mathbb{F}$ is given by

$$\underline{x}_P = \underline{q}_C + R(\underline{q}_R)\underline{r}_P , \tag{43}$$

where R is a 3×3 rotation matrix fixing the orientation of the body with respect to $\mathbb{F}$, while $\underline{r}_P$ is the position vector of point P with respect to the body frame [26]. Then, the unilateral constraint for the contact event examined is expressed in the form

$$\rho(\underline{q}) \equiv s(\underline{x}_P) \geq 0 . \tag{44}$$

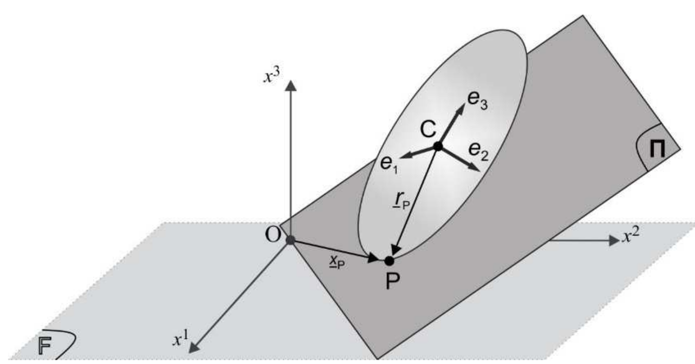

Fig. 4. A rigid body hitting a plane rigid wall.

Using the above expression for the boundary defining function, the first step in the analysis developed in the present study is the determination of the elements $A^1_{I'}$. Based on their definition by Eq. (11), these elements are included in the special covector

$$\underset{\sim}{A}^1 \equiv [A^1_{I'}] = \frac{\partial \rho}{\partial \underline{q}} = (\frac{\partial \rho}{\partial \underline{q}_C} \quad \frac{\partial \rho}{\partial \underline{\vartheta}}) \,. \tag{45}$$

Direct evaluation of these partial derivatives, taking into account Eqs (42)-(44), leads to

$$\frac{\partial \rho}{\partial \underline{q}_C} = (s_1 \quad s_2 \quad s_3) \quad \text{and} \quad \frac{\partial \rho}{\partial \underline{\vartheta}} = -(s_1 \quad s_2 \quad s_3) R \tilde{r}_P \equiv (s_4 \quad s_5 \quad s_6) \,,$$

where $\tilde{r}_P$ is the 3×3 skew-symmetric matrix having $\underline{r}_P$ as an axial vector [26]. This yields

$$\underset{\sim}{A}^1 = (s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6) \,. \tag{46}$$

Moreover, due to the block-diagonal form of the metric matrix in the x-coordinate system, its inverse matrix is also of the same form, with

$$g^{11} = 1/g_{11} \,. \tag{47}$$

Next, combination of Eqs (12) and (13) leads to

$$g^{IJ} = A^I_{I'} A^J_{J'} g^{I'J'} \quad \Rightarrow \quad g^{11} = A^1_{I'} A^1_{J'} g^{I'J'} \,. \tag{48}$$

In addition, if the axes of the body frame are selected to coincide with the principal axes of inertia of the rigid body, the inverse of the metric matrix defined by Eq. (40) is found in the diagonal form

$$G_q^{-1} = [g^{I'J'}] = diag(1/m \quad 1/m \quad 1/m \quad 1/J_{C1} \quad 1/J_{C2} \quad 1/J_{C3}) \,. \tag{49}$$

Therefore, employing Eq. (47), in conjunction with Eqs (46), (48) and (49), it is straightforward to determine the term

$$g_{11} = 1/g^{11} = [(s_1^2 + s_2^2 + s_3^2)/m + s_4^2/J_{C1} + s_5^2/J_{C2} + s_6^2/J_{C3}]^{-1} \,. \tag{50}$$

This, in turn, allows determination of the term $\hat{g}_{11}$, by using Eqs (9) and (10). The quantity $\hat{g}_{11}$ is a key term, since by employing Eq. (14) in conjunction with Eq. (48), results in

$$\hat{g}_{I'J'} = s_{I'} s_{J'} \hat{g}_{11} \,. \tag{51}$$

Subsequently, this leads to evaluation of the terms $\bar{g}_{I'J'}$, through Eqs (15) and (40), which completes the determination of the elements of the metric matrix near the boundary, expressed in the original q-coordinate system.

The determination of the geometric properties needed in the application of Newton's law of motion is completed by next employing Eq. (20), together with Eqs (48) and (49), for evaluating the quantities $B_1^{I'}$, included in the special vector

$$\underline{B}_1 \equiv [B_1^{I'}] = g_{11}(s_1/m \quad s_2/m \quad s_3/m \quad s_4/J_{C1} \quad s_5/J_{C2} \quad s_6/J_{C3})^T \,. \tag{52}$$

Then, using the expression for A^1_{11} and substituting these quantities in Eq. (21) yields the boundary induced terms $\mathrm{A}^{K'}_{I'J'}$ in the form

$$\mathrm{A}^{K'}_{I'J'} = -\tfrac{1}{x^1} s_{I'} s_{J'} s_{L'} g^{L'K'} g_{11} \,,$$

since the contribution of the remaining nonzero terms A^1_{IJ} is negligible. Also, the term $x^1 = \rho(q)$ is evaluated from Eq. (44). Finally, Eq. (17) in conjunction with Eq. (41) complete the determination of the b-affinities $\bar{\Lambda}^{K'}_{I'J'}$ in the q-coordinate system.

At this point, all the information needed for writing the equations of motion either in the local x-coordinate system, by employing Eq. (26), or in the original q-coordinate system, by using Eq. (28), is available.

Synopsis

This study provided a systematic geometric solution to a fundamental problem of Mechanics, referring to dynamics of colliding mechanical bodies during a single frictionless contact event. It was performed within the general framework of Analytical Mechanics and employed some concepts of differential geometry on manifolds with boundary. This boundary was first detected by using the form of the unilateral constraint. Then, an appropriate vector bundle was established for evaluating derivatives on a manifold with boundary. Next, the essential geometric properties needed for the application of Newton's law of motion were determined near the manifold boundary, so that the contact phase takes place close to that boundary. These properties coincide with the usual configuration manifold properties away from the boundary but vary rapidly inside a thin layer, along the normal direction to the boundary. The new approach provided a clear, accurate and global picture on the dynamics during the whole impact phase. Specifically, it was found that the impulse occurs along the direction of the configuration manifold which is normal to the boundary. Moreover, the equations of motion appear in an ODE form, in contrast to classical formulations, which lead to a system of DAEs. In fact, the unilateral constraint alters the geometric properties but does not affect the dimension of the configuration manifold.

The most important contribution of the new approach is that it can describe fully and in a consistent and accurate way the motion of the class of systems examined during the short contact phase. This eliminates the need to consider non-smooth response. Specifically, the velocity remains smooth and its component normal to the boundary vanishes at it, by construction. In fact, it was shown that the figurative point representing the motion of the system in the configuration manifold can not reach the boundary, due to the special action of the boundary. In particular, the presence of the boundary causes a rapid increase in the magnitude of the metric and connection terms related to motion normal to the boundary. In addition to these changes, affecting the inertia properties of the figurative point, a strong spring-like force appears also near the boundary, pushing the figurative point away from it. This force may possess another component, representing the dissipation taking place during impact. These boundary effects assure that the kinetic energy remains bounded. Also, there is no need to assume a loss of indeformability during the contact phase of rigid bodies, which is a common feature of previous studies. Finally, the clarification of the dynamics of the impact process is expected to provide valuable insight and help in the efforts to develop more efficient and robust weak forms and numerical schemes for studying the dynamics and performing control and optimization of mechanical systems involving impacting components.

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