Dynamic bifurcations in slow-fast system of neuronal excitability

Sergey Kirillov*, Vladimir Nekorkin*
Institute of Applied Physics of the Russian Academy of Science, Nizhny Novgorod, Russia

Summary. The mechanisms of spike oscillation generation and suppression in the model of neural excitability with the slowly varying synaptic current are investigated. It was established that the times of occurrence and disappearance of oscillations are determined by dynamic bifurcations and nonlocal threshold properties of two-dimensional invariant surface of the saddle trajectory

When studying the dynamics of nonautonomous systems with slowly varying parameters, one of the basic problems involves the analysis of the mechanisms underlying the transitions between different regimes of activity. Even an arbitrarily small "drift" of control parameter may substantially change system response characteristics compared with quasistatic parameter variation. So for example the transition from rest state to oscillation mode may occur with either significant delay or substantial advance in comparison with critical value obtained by classical bifurcation analysis [1]. To understand the mechanisms of such behavior one should use the theory of so-called dynamic bifurcations, i.e., the bifurcations caused by slow time variations in control parameters [2-4]. Under dynamic bifurcations the phase-space structure of the system changes continuously and transient dynamics is determined by the properties of invariant manifolds.

The recent interest in the effects of slowly varying control parameters arises in neuroscience. The considerable attention to neural systems is due to the fact that the formation of various spatiotemporal structures of activity and switching between them form the basis for perception, analysis and information processing in the brain. The interaction between individual neurons occurs via special contacts (synapses). In many natural experiments the synaptic currents were slowly varying compared with the period of inherent oscillations in neurons.

Here we report the transient dynamics in the model of neural activity under the influence of slow synaptic current [6-8]. We reveal the mechanisms of generation and suppression of spike sequences (the action potential series) in a neuron We study nonlocal oscillation phenomena at dynamic bifurcations and properties of complex dynamic thresholds of neural excitability.

Model

Here we employ the nonautonomous model of neuron firing. The model dynamics is described by modified Fitz-Hugh-Nagumo model that takes into account nonlinear character of ionic recovery currents

$$\begin{aligned} \dot{u} &= f(u) - v + I_{syn}(t), \\ \dot{v} &= \varepsilon\big(g(u) - v\big), \end{aligned} \tag{1}$$

where $f(u) = \gamma u - u^3/3$ is the nonlinear cubic function, the function $g(u)$ has piece-wise linear form given by $g(u) = \alpha u$, for $u \le 0$ and $g(u) = \beta u$, for $u > 0$, $I_{syn}(t)$ is the time-varying synaptic current.

In equation (1) the variable u describes membrane potential dynamics, while the variable v defines membrane ion currents common features. The positive parameter ε defines ion currents rate of change, the positive parameters α, β and γ describe ion currents nonlinear properties.

We consider the case when the synaptic current includes both fast and slow components:

$$I_{syn}(t) = I_{fast}(t) + I_{slow}(t). \tag{2}$$

The fast component of the synaptic current has the form of a short strong pulse whose duration is much less than the average duration of a single spike. The slow component of the synaptic current smoothly changes the depolarization level of the neural membrane on the times comparable to or greater than a single action potential.

We approximate the fast synaptic current incoming at the moment $t = t_0$ by the delta-function:

$$I_{fast}(t) = A\delta(t - t_0), \tag{3}$$

where the modulus of the parameter $|A|$ defines the amplitude of the pulse and its sign sgn(A) defines the type of the pulse (excitatory or inhibitory).

We approximate the slow synaptic current by the slowly monotonically increasing function

$$I_{slow}(t) = \begin{cases} I_0, & \text{if } t < t_0, \\ I_0 + \mu(t - t_0), & \text{if } t_0 \le t \le T_{slow} + t_0, \\ I_m, & \text{if } t > T_{slow} + t_0. \end{cases} \tag{4}$$

where I_0 and I_m defines the depolarization level of the nerve membrane before the activation and after the deactivation of the synaptic current respectively, $0 < |\mu| << 1$ determines the rate of increase of the synaptic current and $T_{slow} = (I_m - I_0)/\mu$ is the time of the action of the synaptic current to the neuron.

Phase space analysis

Let us fix the parameters α, β, ε, and γ by constants, while the parameter I_0 is considered as a control parameter. In this case, depending on the value of the parameter I_0, system (1) may have one to three equilibrium states. For $0 < I_0 < I_0^{SN}$, there exist three equilibrium states in the system: O_1, O_2, and O_3. The equilibrium state O_2 is a saddle point, while O_1 and O_3 are nodes or foci, which can be stable or unstable depending on the value of I_0. The saddle-node bifurcation of the equilibrium states O_2 and O_3 occurs for $I_0 = 0$, while for $I_0 = I_0^{SN}$ the same is observed for the equilibrium states O_1 and O_2. For $I_0 = I_0^{T}$, the cycles C^S and C^U merge, form a two-fold limit cycle, and disappear (the tangent bifurcation of the limit cycles). Another bifurcation, which is undergone by the cycle C^U, occurs for $I_0 = I_0^{H1}$ when it appears from a large loop of the separatrices of the saddle O_2, which is formed by the separatrices W_1^S and W_2^U with increasing parameter I_0. The cycle C^S appears with decreasing parameter I_0 for $I_0 = I_0^{A2}$ as a result of the supercritical Andronov–Hopf bifurcation of the equilibrium state O_3. The appearance and disappearance of the cycle C^U are also related to the bifurcations of the saddle-separatrix loop and the loss of the equilibrium-state stability. The cycle C^U is originated with increasing parameter I_0 for $I_0 = I_0^{H2}$ from a small separatrix loop that is formed by the separatrices W_1^S and W_1^U and disappears for $I_0 = I_0^{A1}$, as a result of the subcritical Andronov–Hopf bifurcation. Note that $0 < I_0^T < I_0^{H1} < I_0^{H2} < I_0^{A1} < I_0^{SN} < I_0^{A2}$.

Let's now consider the case when fast component of synaptic current (2) is absent and the neuron (1) is influenced only by the slow component. For small enough rates $0 < |\mu| << 1$ of current variation the perturbation of the vector field of the model (1) is rather weak in comparison with the autonomous case $\mu = 0$. As the result of such perturbation the so called genuine invariant manifolds flake out from the invariant manifolds of the autonomous system. The basic properties of the genuine invariant manifolds are well described by Fenichel's geometric singular perturbation theory [5]. However this theory becomes inapplicable in two cases. First, when the system undergoes bifurcation and, second, when the motion time on the invariant-manifold is very long. For example, the imaging-point motion along the separatrix of the saddle equilibrium state, occurs for an infinitely long time. In this case, the locally small perturbations of the vector field are accumulated along the phase trajectories, which, in turn, can initiate significant variations in the forms and location of the invariant manifolds with respect to each other.

Generation of spike oscillations

Suppose that (1) is stimulated only with the slow component of increasing synaptic current (I_{fast}=0 and μ>0). Let at the initial time the model is in the rest state ($I_0 < I_0^{A1}$). For this case the dynamic mechanism of the spike generation is found to be based both on the dynamic Andronov–Hopf bifurcation and on the nonlocal oscillation properties in whose formation an important role is played by the behavior of the stable separatrix surfaces of the saddle manifold in the phase space. It is shown that the spike-oscillation appearance is uniquely determined by the depolarization level at the initial time (memory effect) and occurs with the time delay compared with the quasistatic case of the current increase (delay effect). It is found that to find the spike-oscillation appearance time, one should take into account not only the time of slow oscillation phase in the equilibrium-state neighborhood, but also the time which is mainly determined by the nonlocal properties of the solutions of the "fast" equations of the model.

Spike oscillation suppression

Let (1) is stimulated only with the slow component of decreasing synaptic current (I_{fast}=0 and μ<0). Suppose that the model is on the spike oscillation mode ($I_0 > I_0^T$). In this case decreasing synaptic current (μ<0) induces slow passage through the saddle-node bifurcation of the cycles. For this dynamic bifurcation of the cycles, the stable and unstable invariant manifolds neither merge nor disappear as is observed in the classical theory of bifurcations, but continue to exist separately even after the passage through the static bifurcation value. It has been found that rotational motions on the stable invariant manifold, which correspond to the stable oscillation-generation regime, continue to exist for some finite time even after passage through the bifurcation point, i.e., they disappear with a certain time delay. Depending on the control-parameter variation rate, the oscillation-disappearance delay time can be very long and it is determined by the threshold properties of the separatrix surface of the saddle trajectory.

Complex dynamic thresholds

Suppose (1) is influenced both by the fast and by the slow components of the synaptic current. We set the model is in excitable mode ($I_0 < I_0^T$). Then the features of the neuron response are determined by the properties of the complex dynamic threshold of neuron excitability. The structure of the dynamic threshold appearing in the system cannot directly be reconstructed from the static-threshold structure on the assumption of a quasistatic variation of the synaptic current. It has been established that the formation of folds on the threshold manifold whose role is played by the separatrix invariant manifold of the saddle trajectory is a typical feature of the complex dynamic threshold.

References

[1] S.M. Baer, T. Erneux, J. Rinzel, SIAM J. Appl. Math., 1989, 49(1), 55-71.
[2] A.I. Neishtadt, Differential Equations, 1987, 23, 1385-1390.
[3] A.I. Neishtadt, Differential Equations, 1988, 24, 171-176.
[4] E. Benoit (Ed.), Dynamic bifurcations. Springer, Lecture Notes in Mathematics, V. 1493, 1991, 219 p.
[5] N. Fenichel, J. Differential Equations, 1979, 31(1), 53-98.
[6] S.Y. Kirillov, V.I. Nekorkin, Radiophys. Quantum El., 2013, 56(1), 36-50.
[7] S.Y. Kirillov, V.I. Nekorkin, Radiophys. Quantum El., 2014, 57(11), 837-847.
[8] S.Y. Kirillov, V.I. Nekorkin, Radiophys. Quantum El., 2015, 58(12). 951-969.