Bifurcation Analysis of Non-Smooth Floating Bodies

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Summary. This paper examines the stability of a non-smooth floating body, specifically a rectangular prism. A nonlinear model is developed to determine the stability of the upright and tilted equilibria positions as a function of the vertical location of the rigid body's center of mass. These equilibria positions are defined by an angle of rotation and a vertical position where rotational motion is restricted to a two dimensional plane. Numerical investigations are conducted using path following continuation methods to determine equilibria solutions and evaluate stability. Bifurcation diagrams are generated that illustrate the stability of the equilibrium positions as a function of the vertical location of the center of mass. The bifurcation diagrams show complex stability behavior with many coexisting solutions. Experiments are conducted to validate numerical results and an R-Squared value of 0.9987 is observed.

Extended Abstract

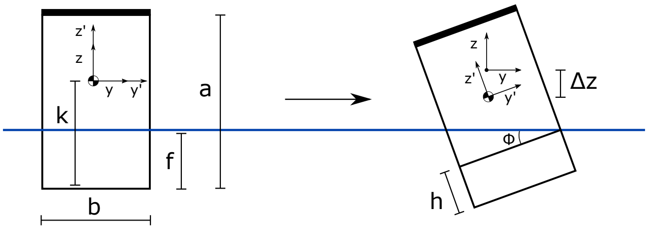


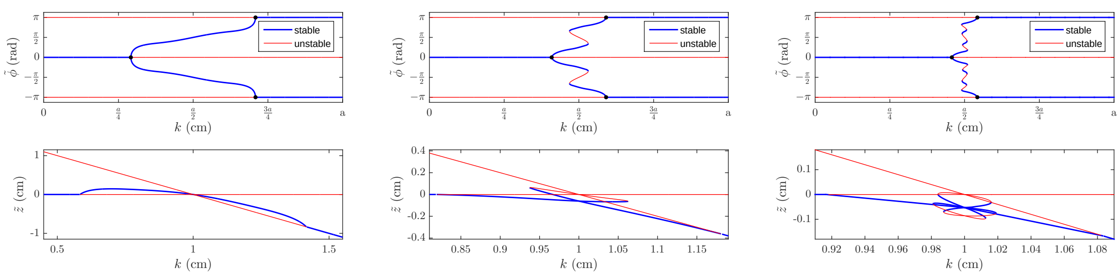
Figure 1: Geometry of a floating rectangular body in both tilted and upright positions with depth ℓ into the page. Planar motion degrees of freedom are assumed where rotation ϕ and vertical translation z give non-trivial equilibrium solutions.

A classic engineering rule of thumb states that a floating object can be made stable if its center of mass is located below the center of mass of the displaced fluid. This fact has been leveraged by engineers for centuries in designing flotation devices that maintain stability under external fluid forces, but loses intuitive usefulness when the object's center of mass is allowed to shift away from its geometric center. As this location is altered, complex geometry dependent behavior develops that result in asymmetric equilibrium orientations that are sensitive to jumps in stability under small perturbations.

This abstract examines the dynamical system characteristics for a non-smooth floating body, specifically a rectangular prism. A nonlinear mathematical model is developed to describe the system which is then used to determine the stability of both upright and tilted equilibria positions as a function of the vertical location of the object's center of mass. Simulations are conducted using time marching algorithms to obtain numerical results that are then validated through experimental studies. This work will enable future studies on the dynamic behavior of discontinuous floating bodies with various real-world applications.

Fig. 1 shows a schematic of the upright and tilted positions of a floating rectangular body along with its defining parameters a, b, ℓ and m. Additionally, k is defined as the vertical distance from its base to the center of mass and is the parameter that will be varied throughout this study. Assuming planar motion, this body's position can be fully described by a rotation, ϕ, and translations, y and z, where y results in trivial solutions for this analysis so will be neglected moving forward.

As this body undergoes rotation and translation, different combinations of vertices become submerged. This results in distinct, piecewise smooth equations of motion where the position to the center of buoyancy with respect to the center of mass, $\vec{R}_{B/G}$, can be calculated by determining the centroid of the submerged volume, V_{sub}, as a sum of multiple volumes.



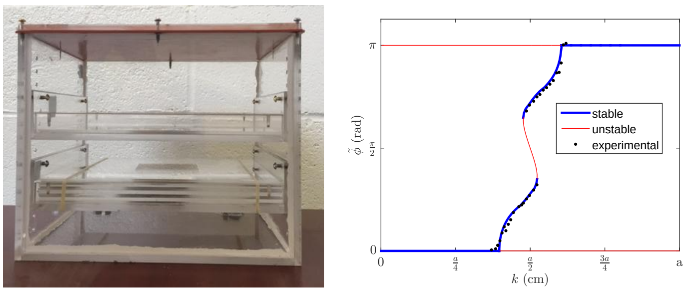
(a) Case 1 (2 coexisting solutions) (b) Case 2 (4 coexisting solutions) (c) Case 3 (8 coexisting solutions)

Figure 2: Bifurcation diagrams for rectangular prisms with varying geometric parameters (a, b, ℓ, and m). Qualitatively different solutions with coexisting equilibria and complex stability behavior are observed by altering these parameters.

Defining $\vec{R}_{B/G} = [0, R_1, R_2]^T$ and $f_b = \rho V_{sub} g$, the equations of motion governing this system can be expressed as

$$\ddot{\phi} = \frac{f_b}{I}\Big[R_1 \cos\phi - R_2 \sin\phi\Big], \qquad \ddot{z} = \frac{1}{m}\Big[f_b - mg\Big].$$

Setting $\ddot{\phi} = \ddot{z} = 0$ and solving for ϕ and z gives the equilibrium solutions, $\tilde{\phi}$ and $\tilde{z}$, for a given set of system parameters. The stability of this position can then be determined by solving the characteristic equation $|A - \lambda I| = 0$, where A is the system's Jacobian evaluated at a given set of equilibria points, and determining when $\mathcal{R}(\lambda) > 0$. These values were obtained as a function of k using numerical continuation methods to produce the bifurcation diagrams shown in Fig. 2 where Figs. 2a, 2b, and 2c were generated for three Cases with varying values of a, b, ℓ, and m.
For all three Cases, the prism has trivial equilibrium solutions for small values of k. These solutions correspond to the prism floating in an upright position and remain stable until k reaches some critical point. These points are labeled on the bifurcation diagrams and can be determined analytically by setting $\lambda = 0$ and solving the characteristic equation to get $k_c = \frac{6m^2+\rho^2 b^4 \ell^2}{12\rho b \ell m}$.
As k is increased beyond k_c, all three Cases undergo supercritical pitchfork bifurcation in $\tilde{\phi}$ producing a branch of solutions that correspond to the prism's tilted equilibrium positions. For Case 1, this branch remains stable until it intersects with the other trivial solution branch π radians from the upright.
In Case 2, this branch undergoes more complicated behavior. At approximately $\frac{\pi}{3}$ radians, this branch encounters a turning point where the solution becomes unstable. At this point, the solution doubles back on itself and remains unstable until approximately $\frac{2\pi}{3}$ radians when it regains stability and doubles back again. From here the solution remains stable until it eventually intersects with the trivial solution at π radians. These turning points result in the system exhibiting metastability for certain ranges of k. This coexistence of stable attractors demonstrates that small changes in k at or near these critical points can cause the system to undergo significant changes in orientation.
Case 3 shows similar behavior to Case 2 but undergoes six turning points instead of just two. As a result, the system in Case 3 is metastable with eight simultaneous stable equilibria points for a range of k. While this range is smaller for Case 3, the implications on the system are the same as discussed for Case 2 with even more possible equilibrium orientations.
While $\tilde{\phi}$ is evolving as a function of k, corresponding changes in $\tilde{z}$ occur. These changes, however, are small in comparison to the scale of the prism and occur only to maintain a constant V_{sub} while the prism is undergoing rotation.



(a) Experimental Setup (b) Results superimposed on predictions

Figure 3: Experimental testing to validate the bifurcation results produced numerically by measuring $\tilde{\phi}$ as a function of varying k.

To validate the numerical results obtained above, an experiment was designed to test the equilibrium positions of a rectangular prism as a function of k. Fig. 3a shows the acrylic prism used for testing where sliding internal weights were used to adjust k without altering the body's mass or geometry. Image processing was used to determine corresponding equilibrium positions where results were reported for only the tilt angle because $\tilde{z}$ was too small to measure accurately in the lab. The geometry tested corresponds to a case which numerical results predicted would exhibit coexisting stable solutions around the point $\tilde{\phi} = \frac{\pi}{2}$. The results from these experiments are superimposed over the bifurcation diagram produced numerically to allow for comparison and are shown in Fig. 3b. These results match well with numerical predictions. As expected, none of the unstable equilibrium positions were able to be measured as evidenced by the jumps in data points around unstable branches. Quantitatively, these results have an R-Squared value of 0.9987 and an RMSE of 0.0675 radians. This agreement constitutes sound validation in the model and gives confidence that the other Cases studied numerically would follow predictions as well.

References

[1] Sequeira D., Mann B.P. (2016) Static Stability Analysis of a Floating Rectangular Prism. ASME 2016 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference. ASME, NC.

[2] Sah S.M., Mann B.P. (2012) Potential well metamorphosis of a pivoting fluid-filled container. *Physica D* **241**:1660-1669.

[3] Strogatz S.H. (2001) Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering. Westview Press, CO.