Differential equations with state-dependent delays — smooth center manifolds and normal forms

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Summary. Delay-differential equations (DDEs) with state-dependent delays suffer from a lack of regularity: even if all coefficients in the equations are apparently smooth, the dependence of the solution on the initial conditions is not continuously differentiable more than once in any of the known choices for phase space. We show that local center-unstable manifolds near equilibria can still be as smooth as the spectral gap in the linearization permits. This result makes many ready-to-use normal form transformation formulas developed by Govaerts, Kuznetsov *et al* for ordinary differential equations (recently extended to DDEs with constant delays) applicable to DDEs with state-dependent delays. We give a first demonstration how normal form coefficients can be obtained for a Hopf bifurcation in a scalar DDE with several nesting levels for the delays.

Delay-differential equations with state-dependent delays

The general form for a delay-differential equation (DDE) with (bounded) state-dependent delay (abbreviated here sd-DDEs) is

$$\dot{x}(t) = f(x_t, p) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the current state, $p \in \mathbb{R}^{n_p}$ are the system parameters, f maps $C([-\tau_{\max}, 0]; R^n) \times \mathbb{R}^{n_p}$ into $\mathbb{R}^n$, and we use the notation $x_t(\theta) = x(t+\theta)$. ($C^0 := C([-\tau_{\max}, 0]; R^n)$ is the space of continuous functions on $[-\tau_{\max}, 0]$ with values in $\mathbb{R}^n$. Similarly C^K will be the space of k times continuously differentiable functions.) Hence, the evolution of (1) needs a function segment on $[-\tau_{\max}, 0]$ as its initial value. Examples for models where state-dependent delays occur are blood-cell formation models [1], models for regenerative cutting when the tool has finite stiffness in the direction tangential to the rotation direction of the work piece [4], or time-delayed feedback control with continuous adjustment of the time delay [7] (see Hartung *et al* [3] for a review of applications and theory of sd-DDEs up to 2006).
Whether the delay is state dependent can inferred from (or even defined by) a lack of regulatiy of the right-hand side f. If $f : C([-\tau_{\max}, 0]; \mathbb{R}^n) \times \mathbb{R}^{n_p} \mapsto \mathbb{R}^n$ is continuously differentiable with respect to its first argument we speak of a DDE with *constant* delays. To illustrate this problem, let us consider the simple equation

$$\dot{x}_t = p - x(t - x(t)) \tag{2}$$

for $p \approx \pi/2$. The functional f for (2) is $f(x, p) = p - x(-x(0))$, which is well defined for x close to $\pi/2$ (that is, $\|x - \pi/2\|_\infty := \max\{|x(\theta) - \pi/2| : \theta \in [-\tau_{\max}, 0]\} \ll 1$ with, for example, $\tau_{\max} = \pi$). The kth derivative of f exists only if x is itself k times differentiable. For example, the first partial derivative of f is

$$\partial_1 f(x,p)y = -y(-x(0)) + x'(-x(0))y(0) \quad \text{(using prime for the derivative of } x \text{ with respect to its argument).}$$

However, trajectories of (2) (and generally (1)) are not k times differentiable for small t. Thus, the dependence of the solution x_t at times $t > 0$ on the initial condition x_0 is not differentiable if we choose C^0 as our phase space (the solution is not even unique, see [3]). Walther proved that the solution x_t of a sd-DDE $\dot{x}(t) = f(x_t)$ (without parameter) depends continuously differentiable *once* on x_0 on the manifold $D_1 := \{x \in C^1([-\tau_{\max}, 0]; \mathbb{R}^n) : x'(0) = f(x)\}$ if f is continuously differentiable as a map from $C^1 \mapsto \mathbb{R}^n$, if the derivative $\partial f(x)$ can be applied to continuous linear deviations $y \in C^0$, and if the map $(x, y) \in C^1 \times C^0 \mapsto \partial f(x)y \in \mathbb{R}^n$ is continuous [12]. For the example (2) Walther's condition means that the derivative $\partial_1 f(x,p)y = -y(-x(0)) + x'(-x(0))y(0)$ can depend on x' but not on y' (only on y). This is the best known result for dependence of the solution x_t on its initial value x_0. In particular, restricting the phase space further (to subsets of D_1 with higher order of differentiability and possibly more constraints) does not help to gain regularity of the solution map.

Bifurcation theory

The situation is better when one studies invariant sets such as equilibria, periodic orbits or finite-dimensional invariant manifolds. Computation of equilibria and their non-dynamic bifurcations (such as saddle-node bifurcations) follows exactly that of ODEs (solving the smooth algebraic system $f(x_{\text{eq}}, p) = 0$, where x_{eq} is a constant function). The stability of an equilibrium x_{eq} is determined by the linear DDE $\dot{y}(t) = \partial_1 f(x_{\text{eq}}, p)y_t$ with constant coefficients [3].
Similarly, periodic orbits, as solutions of periodic boundary value problems (BVPs), can be determined as solutions of finite-dimensional smooth algebraic systems of equations such that all computations typically performed in numerical bifurcation analysis software (such as DDE-Biftool [2, 9] or knut [8]) are well posed. Stability of periodic orbits is again determined by a linear DDE with periodic coefficients [6]. The following theorem covers local bifurcation theory near equilibria more generally by stating that near an equilibrium there exists a local centre-unstable manifold, the smoothness of which is only limited by the spectral gap.
Let $f : C^0([-\tau_{\max}, 0]; \mathbb{R}^n) \mapsto \mathbb{R}^n$ be continuous such that f restricted to C^k is k times continuously differentiable and that its kth derivative can be extended to multi-linear deviations in C^{k-1} continuously (similar to Walther's condition

for the first derivative). We assume that $f(0) = 0$ (putting the equilibrium at the origin and dropping the parameter p without loss of generality) and that $\partial f(0)$ has a *spectral gap*, that is, it has no eigenvalues with real part in $[-\lambda_s, -\lambda_c]$ (where $0 < \lambda_c < \lambda_s$). Let $n_c < \infty$ be the dimension of the invariant subspace of $\partial f(0)$ in C^0 corresponding to the eigenvalues with real part greater than $-\lambda_c$, and $A_c \in \mathbb{R}^{n_c \times n_c}$ be the restriction of $\partial f(0)$ to this subspace. Then there exists a graph $G : \mathbb{R}^{n_c} \mapsto C^0$ such that solutions of $\dot{x}(t) = f(x_t)$ starting in the image $\operatorname{rg} G$ of G stay in $\operatorname{rg} G$ for all times $t \in \mathbb{R}$ for which they stay small. If $x_0 = G(x_c^0)$ then $x_t = G(x_c(t))$, where $x_c(t)$ satisfies a n_c-dimensional ODE $\dot{x}_c(t) = A_c x_c(t) + g(x_c(t))$, $x_c(0) = x_c^0$. The above existence result was already obtained in [10]. However,

G (and, hence, g) is k times continuously differentiable if $k < \lambda_s/\lambda_c$.

When one includes this smoothness statement, the conclusions for the centre-unstable manifold G of the sd-DDE are as strong as they are in the case for ODEs. The proof of the theorem uses the graph transform, but does not require differentiability of the (semi-)flow in arbitrary elements of the phase space.

The result permits us to transfer methods for automatic normal form computations to sd-DDEs. These methods have recently been introduced to DDEs with constant delays in [5, 11], but they assume the existence of a smooth centre manifold.

As an illustrative example, we analyze a generalization of (2) using DDE-Biftool. Consider the equation

$$\dot{x}(t) = f(x_t, p) = p - x(t - x(t - \ldots x(t - x(t))\ldots)) \quad \text{with } k \text{ levels of nesting.} \tag{3}$$

DDE-Biftool [9] permits bifurcation analysis of this type of equation, including continuation of periodic orbits and their bifurcations. A modification of the normal form extension for equilibria (originally by B. Wage [11] for constant delays) can now determine the stability of the Hopf bifurcation of $x = p$ at $p = \pi/2$ for arbitrary nesting levels k in (3). Figure 1 shows the families of periodic orbits branching off and the corresponding Lyapunov coefficients determining stability.

```
%% DDE-Biftool code nested_minimal (Matlab)
clear
ntau=2;           % chosen level of nesting
funcs=set_funcs(...
    'sys_rhs',@(x,p)p-x(1,ntau+1,:),...
    'sys_ntau',@()ntau,...
    'sys_tau',@(nr,x,p)x(1,nr,:),...
    'x_vectorized',true);
eqbr=SetupStst(funcs,'contpar',1,'x',pi/2,...
    'parameter',pi/2-1e-2,'step',0.1);
hopf=SetupHopf(funcs,eqbr,1);
hpl1=nmfm_func_hopf(funcs,hopf.point); % new
display(hpl1.nmfm.L1,'L1');

>> nested_minimal
L1 =
      -0.84286
```

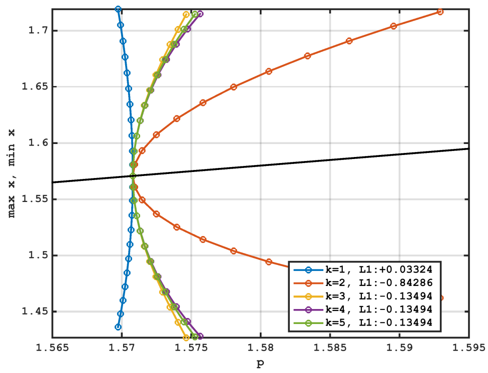

Figure 1: Lyapunov coefficients and families of periodic orbits for (3) at Hopf bifurcation at $p = \pi/2$, $x = p$.

References

[1] M. Craig, A.R. Humphries, and M.C.Mackey. A mathematical model of granulopoiesis incorporating the negative feedback dynamics and kinetics of G-CSF/neutrophil binding and internalisation. *Bulletin of Mathematical Biology*, 78:2304–2357, 2016.

[2] K. Engelborghs, T. Luzyanina, and D. Roose. Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL. *ACM Transactions on Mathematical Software*, 28(1):1–21, 2002.

[3] F. Hartung, T. Krisztin, H.-O. Walther, and J. Wu. Functional differential equations with state-dependent delays: theory and applications. In P. Drábek, A. Cañada, and A. Fonda, editors, *Handbook of Differential Equations*: *Ordinary Differential Equations*, volume 3, chapter 5, pages 435–545. North-Holland, 2006.

[4] T. Insperger, G. Stépán, and J. Turi. State-dependent delay model for regenerative cutting processes. In *Proceedings of the Fifth EUROMECH Nonlinear Dynamics Conference, Eindhoven, Netherlands*, 2005.

[5] S. Jaanssens. On a normalization technique for codimension two bifurcations of equilibria of delay differential equations. Master's thesis, Utrecht University (NL), 2010. supervised by Yu.A. Kuznetsov and O. Diekmann.

[6] J. Mallet-Paret and R. D. Nussbaum. Stability of periodic solutions of state-dependent delay-differential equations. *Journal of Differential Equations*, 250(11):4085 – 4103, 2011.

[7] V. Pyragas and K. Pyragas. Adaptive modification of the delayed feedback control algorithm with a continuously varying time delay. *Physics Letters A*, 375(44):3866–3871, 2011.

[8] D. Roose and R. Szalai. Continuation and bifurcation analysis of delay differential equations. In B Krauskopf, H M Osinga, and J Galán-Vioque, editors, *Numerical Continuation Methods for Dynamical Systems*: *Path following and boundary value problems*, pages 51–75. Springer-Verlag, Dordrecht, 2007.

[9] J. Sieber, K. Engelborghs, T. Luzyanina, G. Samaey, and D. Roose. *DDE-BIFTOOL Manual — Bifurcation analysis of delay differential equations*.

[10] E. Stumpf. The existence and c^1-smoothness of local center-unstable manifolds for differential equations with state-dependent delay. *Rostocker Mathematisches Kolloquium*, 66:3–44, 2011.

[11] B. Wage. Normal form computations for delay differential equations in DDE-Biftool. Master's thesis, Utrecht University (NL), 2014. supervised by Y.A. Kuznetsov.

[12] H.-O. Walther. Smoothness properties of semiflows for differential equations with state-dependent delays. *Journal of Mathematical Sciences*, 124(5):5193–5207, 2004.