On the mathematical justification of viscoelastic shell models

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Summary. We consider a family of linearly viscoelastic shells of thickness 2ε and we use asymptotic analysis to study the limit behaviour of the three-dimensional problems and their solutions when ε tends to zero. Then, depending on the order of the applied forces, the geometry of the middle surface and the set where the shell is clamped, we find two sets of limit two-dimensional equations, namely, the viscoelastic membrane or flexural problems. In both cases, we find a model which presents a long-term memory that takes into account the deformations at previous times. We provide the existence and uniqueness of solution and convergence results.

Introduction

In the last decades, many authors have applied the asymptotic methods in the three-dimensional elasticity problems to derive new reduced one-dimensional or two-dimensional models and justify the existing ones. A complete theory regarding elastic shells can be found in [1], where models for elliptic membranes, generalized membranes and flexural shells are presented. It contains a full description of the asymptotic procedure that leads to the corresponding sets of two-dimensional equations. More recently in [2] the obstacle problem for an elastic elliptic membrane has been identified and justified as the limit problem for a family of unilateral contact problems for elastic elliptic shells. A large number of actual physical and engineering problems have made it necessary the study of models which take into account effects such as hardening and memory of the material. An example of these are the viscoelastic models (see for example [3, 4]). In some of these models, we can find terms which take into account the history of previous deformations or stresses of the body, hence the reference to long-term memory. For a family of shells made of a long-term memory viscoelastic material we can find in [5] the use of asymptotic analysis to obtain the limit two-dimensional membrane and flexural equations. In this work, we analyse the asymptotic behaviour of the scaled three-dimensional displacement field of a shell made of a short-term memory viscoelastic material, as the thickness approaches zero. We consider that the displacements vanish in a portion of the lateral face of the shell, obtaining the equations of a viscoelastic membrane shell or of a viscoelastic flexural shell depending on the order of the forces and the geometry. We will follow the notation and style of [1], where the linear elastic shells are studied.

The three-dimensional linearly viscoelastic shell problem

In what follows, we will use summation convention on repeated indices. Moreover, Latin indices $i, j, k, l, ...$, take their values in the set $\{1, 2, 3\}$, whereas Greek indices $\alpha, \beta, \sigma, \tau, ...$, do it in the set $\{1, 2\}$. Also, we use standard notation for the Lebesgue and Sobolev spaces. Also, for a time dependent function u, we denote $\dot{u}$ the first derivative of u with respect to the time variable. Let ω be a domain of $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\gamma = \partial\omega$. Let $\boldsymbol{y} = (y_\alpha)$ be a generic point of its closure $\bar{\omega}$ and let ∂_α denote the partial derivative with respect to y_α.
Let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha \boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}(\boldsymbol{y})$. We can consider the two vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$ of the same tangent plane defined by the relations $\boldsymbol{a}^\alpha(\boldsymbol{y}) \cdot \boldsymbol{a}_\beta(\boldsymbol{y}) = \delta^\alpha_\beta$, that constitute the contravariant basis. We define the unit vector, $\boldsymbol{a}_3(\boldsymbol{y}) = \boldsymbol{a}^3(\boldsymbol{y}) := \frac{\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})}{|\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})|}$, normal vector to S at the point $\boldsymbol{\theta}(\boldsymbol{y})$, where $\wedge$ denotes vector product in $\mathbb{R}^3$. We can define the first fundamental form, given as metric tensor, in covariant or contravariant components, respectively, by $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$, $a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$, the second fundamental form, given as curvature tensor, in covariant or mixed components, respectively, by $b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha$, $b^\beta_\alpha := a^{\beta\sigma} \cdot b_{\sigma\alpha}$, and the Christoffel symbols of the surface S by $\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha$. The area element along S is $\sqrt{a}dy$ where $a := \det(a_{\alpha\beta})$.
Let γ_0 be a subset of γ, such that $meas(\gamma_0) > 0$. For each $\varepsilon > 0$, we define the three-dimensional domain $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ and its boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$. We also define the parts of the boundary, $\Gamma^\varepsilon_+ := \omega \times \{\varepsilon\}$, $\Gamma^\varepsilon_- := \omega \times \{-\varepsilon\}$ and $\Gamma^\varepsilon_0 := \gamma \times [-\varepsilon, \varepsilon]$.
Let $\boldsymbol{x}^\varepsilon = (x^\varepsilon_i)$ be a generic point of $\bar{\Omega}^\varepsilon$ and let ∂^ε_i denote the partial derivative with respect to x^ε_i. Note that $x^\varepsilon_\alpha = y_\alpha$ and $\partial^\varepsilon_\alpha = \partial_\alpha$. Let $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ be the mapping defined by $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) := \boldsymbol{\theta}(\boldsymbol{y}) + x^\varepsilon_3 \boldsymbol{a}_3(\boldsymbol{y}) \, \forall \boldsymbol{x}^\varepsilon = (\boldsymbol{y}, x^\varepsilon_3) = (y_1, y_2, x^\varepsilon_3) \in \bar{\Omega}^\varepsilon$. Furthermore for $\varepsilon > 0$, $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon) := \partial^\varepsilon_i \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ are linearly independent and hence, the three vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)$ form the covariant basis of the tangent space at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ and $\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$ defined by the relations $\boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^\varepsilon_j = \delta^i_j$ form the contravariant basis at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. We define the metric tensor, in covariant or contravariant components, respectively, by $g^\varepsilon_{ij} := \boldsymbol{g}^\varepsilon_i \cdot \boldsymbol{g}^\varepsilon_j$, $g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}$, and Christoffel symbols by $\Gamma^{p,\varepsilon}_{ij} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial^\varepsilon_i \boldsymbol{g}^\varepsilon_j$.
The volume element in the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$ is $\sqrt{g^\varepsilon} dx^\varepsilon$ and the surface element in $\boldsymbol{\Theta}(\Gamma^\varepsilon)$ is $\sqrt{g^\varepsilon} d\Gamma^\varepsilon$ where $g^\varepsilon := \det(g^\varepsilon_{ij})$.
Besides, let $T > 0$ be the time period of observation and we denote by $u^\varepsilon_i : [0, T] \times \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ the covariant components of the displacements field, that is $\boldsymbol{\mathcal{U}}^\varepsilon := u^\varepsilon_i \boldsymbol{g}^{i,\varepsilon} : [0, T] \times \bar{\Omega}^\varepsilon \to \mathbb{R}^3$. We assume that the shell is subjected to a boundary condition of place; in particular, the displacements field vanishes in $\boldsymbol{\Theta}(\Gamma^\varepsilon_0)$, this is, a portion of the lateral face of the shell. We consider that the body is made of a Kelvin-Voigt viscoelastic material, which is homogeneous and isotropic, so that the material is characterized by its Lamé coefficients $\lambda \geq 0, \mu > 0$ and its viscosity coefficients, $\theta \geq 0, \rho \geq 0$ (see for

instance [3, 4]), independent of ε. Under the effect of applied forces, the body is deformed and we can find that $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ verifies the following variational problem of a three-dimensional viscoelastic shell in curvilinear coordinates:

Problem 1 *Find* $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ *such that:* $\boldsymbol{u}^\varepsilon(t,\cdot) \in V(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3; \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon\}\ \forall\, t \in [0,T]$,

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} B^{ijkl,\varepsilon} e^\varepsilon_{k||l}(\dot{\boldsymbol{u}}^\varepsilon) e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon$$
$$= \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} h^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon} d\Gamma^\varepsilon \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon),\ a.e.\ in\ (0,T), \quad \boldsymbol{u}^\varepsilon(0,\cdot) = \boldsymbol{u}_0^\varepsilon(\cdot),$$

where the functions $A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon}g^{jl,\varepsilon} + g^{il,\varepsilon}g^{jk,\varepsilon})$, $B^{ijkl,\varepsilon} := \theta g^{ij,\varepsilon}g^{kl,\varepsilon} + \frac{\rho}{2}(g^{ik,\varepsilon}g^{jl,\varepsilon} + g^{il,\varepsilon}g^{jk,\varepsilon})$, are the contravariant components of the three-dimensional elasticity and viscosity tensors, respectively. Moreover, the terms $e^\varepsilon_{i||j}(\boldsymbol{u}^\varepsilon) := \frac{1}{2}(u^\varepsilon_{i||j} + u^\varepsilon_{j||i}) = \frac{1}{2}(\partial^\varepsilon_j u^\varepsilon_i + \partial^\varepsilon_i u^\varepsilon_j) - \Gamma^{p,\varepsilon}_{ij} u^\varepsilon_p$, designate the covariant components of the linearized strain tensor associated with the displacement field $\boldsymbol{\mathcal{U}}^\varepsilon$of the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$. Besides, $f^{i,\varepsilon}$ and $h^{i,\varepsilon}$ denote the contravariant components of the volumic and surface force densities, respectively and $\boldsymbol{u}_0^\varepsilon$ denotes the initial " displacements ".
We prove the existence and uniqueness of solution of the Problem 1 for $\varepsilon > 0$ small enough. Then, under suitable regularity hypotheses for the applied forces and initial condition then $\boldsymbol{u}^\varepsilon \in W^{1,2}(0,T; V(\Omega^\varepsilon))$.

Formal Asymptotic Analysis. Obtention of the two-dimensional limit problems

In order to perform the asymptotic analysis we write Problem 1 defined over a reference domain independent of ε. Then, we assume that there exists an asymptotic expansion of the unknown and its initial condition and we substitute them into the equations. Considering different order for the applied forces we are able to identify terms of the asymptotic expansion proposed. In particular, we find in [6] that its main leading term is independent of the transversal variable x_3. Therefore, it can be identified with a function $\boldsymbol{\xi} \in [H^1(\omega)]^3$ such that $\boldsymbol{\xi} = \boldsymbol{0}$ on γ_0. Then, we find that $\boldsymbol{\xi}$ is solution of two different sets of equations depending on whether or not the space $V_0(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in V(\omega), \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$, contains non-zero functions. Furthermore, considering the right spaces where these problems are well posed, we obtain that the de-scaled function $\boldsymbol{\xi}^\varepsilon$ is the unique solution of what we have identified as the viscoelastic membrane or viscoelastic flexural problem, respectively. For instance, for the viscoelastic membrane case we obtain the following limit two-dimensional problem:

Problem 2 *Find* $\boldsymbol{\xi}^\varepsilon$ *such that:* $\boldsymbol{\xi}^\varepsilon(t,\cdot) \in V_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)\ \forall\, t \in [0,T]$,

$$\varepsilon \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\xi}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy + \varepsilon \int_\omega b^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\dot{\boldsymbol{\xi}}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy$$
$$- \varepsilon \int_0^t e^{-k(t-s)} \int_\omega c^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\xi}^\varepsilon(s))\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dyds$$
$$= \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}dy,\ \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega),\ a.e.\ in\ (0,T), \quad \boldsymbol{\xi}^\varepsilon(0,\cdot) = \boldsymbol{\xi}_0^\varepsilon(\cdot),$$

where, $\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$ *denote the covariant components of the change of metric tensor associated with a displacement* $\eta_i\boldsymbol{a}^i$ *of* S, $p^{i,\varepsilon}(t) := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}(t)dx_3^\varepsilon + h_+^{i,\varepsilon}(t) + h_-^{i,\varepsilon}(t)$ *and* $h_\pm^{i,\varepsilon}(t) = h^{i,\varepsilon}(t,\cdot,\pm\varepsilon)$ *and where the contravariant components of the fourth order two-dimensional tensors* $a^{\alpha\beta\sigma\tau,\varepsilon}$, $b^{\alpha\beta\sigma\tau,\varepsilon}$, $c^{\alpha\beta\sigma\tau,\varepsilon}$ *arised naturally in the asymptotic analysis performed.*

We prove the existence and uniqueness of solution of Problem 2. Under suitable regularity for the applied forces and initial condition then $\boldsymbol{\xi}^\varepsilon \in W^{1,2}(0,T; V_M(\omega))$. We also provide convergence theorems that justify the equations obtained.

Conclusions

We have found and justified limit two-dimensional models for viscoelastic membrane shells and viscoelastic flexural shells. To this end we used the asymptotic expansion method to identify the variational equations from the scaled three-dimensional viscoelastic shell problem. The main novelty is that from the asymptotic analysis of the three-dimensional problems which include a short-term memory represented by a time derivative, a long-term memory arises in the two-dimensional limit problems, represented by an integral with respect to the time variable.

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