Oscillation Patterns in Stochastic Fast-Slow Systems

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Summary. The effect of Gaussian white noise on fast-slow stochastic differential equations (SDEs) can induce, modify or destroy deterministic oscillations. One key example discussed in this contribution are fast-slow systems exhibiting mixed-mode oscillations owing to the presence of a folded-node singularity. The stochastic system can be described by a continuous-space, discrete-time Markov chain, recording the returns of sample paths to a Poincaré section. We provide estimates for this Markov chain, depending on the system parameters and the noise intensity based upon results for local noisy folded singularities and global decomposition. The results yield predictions on the observed random mixed-mode oscillation patterns. There is an intricate interplay between the number of small-amplitude oscillations and the global return mechanism. Furthermore, we shall outline that the methods to analyze the MMOs generalize to many different classes of fast subsystem bifurcations. This relates the theory to early-warning signs for critical transitions, to various applications in mathematical biology and to numerical methods for stochastic systems.

Introduction

Oscillations are a frequent phenomenon in a wide variety of nonlinear dynamical systems. Systems with multiple time scales are known to exhibit several interesting generic oscillation patterns. Examples are relaxation oscillations, which consist of simple alternating fast and slow motion segments. A classical example exhibiting relaxation oscillations is the van der Pol equation

$$\begin{array}{rcl} \varepsilon\frac{\mathrm{d}x}{\mathrm{d}\tau} &=& y-\frac{1}{3}x^3+x, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &=& -x, \end{array} \tag{1}$$

where $(x,y)=(x(\tau),y(\tau))\in\mathbb{R}^2$ are the phase space variables, $\tau\in\mathbb{R}$ is the time variable, and $\varepsilon>0$ is a small parameter indicating the separation of time scales in the fast-slow system (1). Note that in (1), the variable x is fast and the variable y is slow. If one goes to higher-dimensional phase spaces, several other oscillation patterns are possible beyond classical relaxation oscillations. For example, one frequently observes bursting oscillations as well as mixed-mode oscillations (MMOs). Mixed-mode oscillations are characterized by a separation into large-amplitude oscillations (LAOs) and small-amplitude oscillations (SAOs) within a time series of a single periodic trajectory. These complex MMOs are usually generated by an interplay of two crucial ingredients. The first one is a small-scale mechanism near certain singularities or unstable steady state solutions, where fast and slow variables have, for a short period, temporally comparable speeds in their evolution. This can lead to SAOs. The second ingredient is a global return mechanism for the dynamics, which allows one to get back repeatedly to a certain region in phase space after a long excursion. This aspect induces the SAOs.

The Stochastic Koper Model

The analysis regarding effects of noise on oscillation patterns is still developing. The motivation to study stochastic differential equations (SDEs) instead of ordinary differential equations (ODEs) is very clear: random terms enter the modelling in very natural ways, e.g., via external random forcing, via intrinsic finite system-size noise in approximations of particle systems, via uncertain parameters, or as modelling uncertainties. However, the mathematical analysis of fast-slow stochastic systems becomes substantially more complicated. In this contribution, we shall focus on several results obtained in the context of a relatively broad class of stochastic fast-slow systems exhibiting MMOs. A simple, yet quite typical, representative of this class is the stochastic Koper model with additive noise:

$$\begin{array}{rcl} \varepsilon\frac{\mathrm{d}x}{\mathrm{d}\tau} &=& y-x^3+3x+\sigma_1\xi_1, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &=& kx-2(y+\lambda)+z+\sigma_2\xi_2, \\ \frac{\mathrm{d}z}{\mathrm{d}\tau} &=& \lambda+y-z+\sigma_3\xi_3 \end{array} \tag{2}$$

where $(x,y,z)=(x(\tau),y(\tau),z(\tau))\in\mathbb{R}^3$ are now the phase space variables, $\xi_j=\xi_j(\tau)$ for $j\in\{1,2,3\}$ are independent white noises, σ_j are parameters to control the noise level, and λ,k are the main bifurcation parameters. Recall that formally we have mean zero and delta-correlation for the white noises, i.e., $\mathbb{E}[\xi_j(\tau)]=0$ for any $\tau\in\mathbb{R}$ and $\mathbb{E}[\xi_j(\tau)\xi_j(s)]=\delta(\tau-s)$. Mathematically, we can view ξ_j as a generalized stochastic process or re-write (2) as an Itô or Stratonowich SDE. Observe carefully that one may view the deterministic version of (2) for $\sigma_1=0$, $\sigma_2=0$ and $\sigma_3=0$ as a natural generalization to three variables of the classical van der Pol oscillator (1). In fact, the Koper model has not only been derived by Koper but independently by many groups around the world as a simple standard model for MMOs.

Sketch of Results

SODEs with MMOs, such as the stochastic Koper model, present several difficulties. Technical problems arise as one has to control a stochastic processes, dynamical problems appear as new phenomena can be realized by the stochastic forcing, and new numerical challenges appear as certain deterministic schemes well-known for nonlinear dynamical systems analysis do not have straightforward generalizations. In this contribution we shall mainly focus on the effect that noise can have on MMOs. We cannot state the technical results here in full detail but just outline a few interesting aspects. First of all, the noise can influence SAOs as it has also triggers small-scale oscillations. Results will be presented, which show how small scale deterministic oscillations can, or cannot, be distinguished from noise. The next key step to be presented is, how to control the stochastic process more globally. We consider recording returns of trajectories to a cross-section, which extends the classical approach via Poincaré maps. In particular, we give probability estimates, how likely it is to return to a certain subset of the section. This requires careful control of the global dynamics under the influence of noise. The approach will be based upon analyzing different phases of the sample paths and provide upper bounds via covariance ellipsoids adapted to the fast-slow structure of the dynamical system. This shows that noise can alter, enhance or diminish deterministic MMOs. Therefore, including stochastic terms does have very important practical consequences. Furthermore, we are going to illustrate our results via numerical simulations, including a view towards specialized algorithms to deal with the dynamical systems analysis of (fast-slow) SODEs.

Literature Summary

The contribution presents only one aspect in a series of results obtained recently in several works on stochastic fast-slow systems. Here we provide a very brief overview of this literature, to which the author of this work has contributed. The literature list also contains some pointers to background results and extensions:

- General background on fast-slow systems [Kue15].
- Background on mixed-mode oscillations in fast-slow systems [DGK$^+$12].
- Decomposition of MMOs into different phase maps for the Koper model [Kue11b].
- Local analysis near folded singularities for SDEs [BGK12].
- Analysis of MMO patterns in three-dimensional normal form SDEs [BGK15].
- Background on critical transitions and warning signs [Kue11a].
- Scaling laws and applications for warning-signs up to codimension two [Kue13].
- Numerical continuation for stochastic oscillating systems [Kue12].
- Bounded noise, early-warning signs, return maps and MMOs [KMR15]

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