

## Energy exchange and localization in essentially nonlinear oscillatory systems: canonical formalism.

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*Summary.* Over recent years, a lot of progress has been achieved in understanding of the relationship between localization and transport of energy in essentially nonlinear oscillatory systems. In this presentation we are going to demonstrate that the structure of the resonance manifold can be conveniently described in terms of canonical action-angle variables. Such formalism has important theoretical advantages: all resonance manifolds may be described at the same level of complexity, appearance of additional conservation laws on these manifolds is easily proven both in autonomous and non-autonomous settings. Application of the canonic averaging allows treatment of much broader variety of dynamical models than those amenable to common harmonic balance-based methods. As an example, energy exchanges in models of coupled vibro-impact oscillators are considered.

### Theoretical framework

Let us consider the conservative system of two coupled oscillators. Generally speaking, the Hamiltonian of this system is expressed as:

$$H = H(p_1, p_2, q_1, q_2) \quad (1)$$

Here  $q_k, k=1,2$  are generalized coordinates, and  $p_k, k=1,2$  are conjugate momenta. It is supposed that at given energy level System (1) occupies a finite hypersurface fragment in the state space.

Each particular Hamiltonian  $H_0(p, q)$  generically induces canonical transformation to action-angle (AA) variables [1, 2]. For each conjugate pair of variables in Hamiltonian (1) we use one of such transformations:

$$p_k = p_k(I_k, \theta_k), \quad q_k = q_k(I_k, \theta_k), \quad k=1,2; \quad I_k \in [0, \infty), \quad \theta_k \in [0, 2\pi) \quad (2)$$

It is not required that the transformations for different  $k$  will be induced by the same Hamiltonian and will have the same functional form. As a result of the transformation, the system will be described by the following Hamiltonian in terms of the action – angle variables, with subsequent Fourier expansion:

$$H(I_1, I_2, \theta_1, \theta_2) = \sum_{m,n} V_{m,n}(I_1, I_2) \exp(i(m\theta_1 - n\theta_2)), \quad V_{m,n} = V_{-m,-n}^* \quad (3)$$

Averaging procedures in Hamiltonians similar to (3) are always based on existence of slowly varying combination of the angle variables. Commonly, this slow phase exists due to the fact that the actions do not deflect much from their average values [3, 4]. It will be demonstrated below that the slow phase may appear also due to other reasons. At this stage, we suppose that the phase variables combine into a single slow phase  $\mathcal{G} = m_0\theta_1 - n_0\theta_2, m_0, n_0 \in \mathbb{Z}$ . Averaging the Hamiltonian over the fast phases, one just removes from (3) all terms not proportional to the slow phase, substitutes the actions by their average values and then obtains a slow-flow Hamiltonian in the following form:

$$\bar{H}(J_1, J_2, \mathcal{G}) = \sum_l V_{m_0 l, n_0 l}(J_1, J_2) \exp(il(m_0\theta_1 - n_0\theta_2)) = \sum_l V_{m_0 l, n_0 l}(J_1, J_2) \exp(il\mathcal{G}), \quad J_k = \langle I_k \rangle, \quad k=1,2, \quad l \in \mathbb{Z} \quad (4)$$

It is possible to prove that the average system possesses additional integral of motion:

$$n_0 J_1 + m_0 J_2 = N^2 = \text{const} \quad (5)$$

Performing trigonometric change of variables  $J_1 = N^2 \sin^2(\gamma/2) / n_0, J_2 = N^2 \cos^2(\gamma/2) / m_0$ , one finally arrives to integrable dynamical system on a sphere with the following first integral:

$$h(\gamma, \mathcal{G}) = \bar{H}\left(\frac{N^2 \sin^2(\gamma/2)}{n_0}, \frac{N^2 \cos^2(\gamma/2)}{m_0}, \mathcal{G}\right) = \text{const}, \quad \mathcal{G} \in [0, 2\pi), \quad \gamma \in [0, \pi] \quad (6)$$

### Example – linearly coupled impact oscillators

We consider the strongest possible nonlinearity and consider a pair of identical impact oscillators, coupled by a linear spring of stiffness  $\varepsilon$  (see Figure 1).

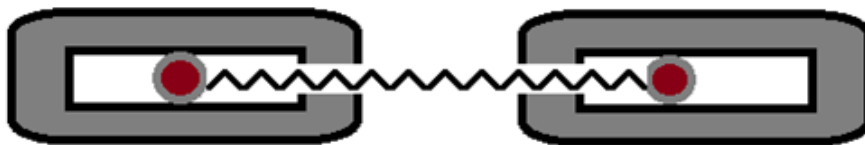


Figure 1. Pair of impact oscillators coupled by linear spring.

In Figure 2 we present the results of the simulation for the system depicted in Figure 1. Initially both impactors are located at the middle points of the channels, i.e.  $u_1(0) = u_2(0) = 0$ ; initial velocity of impactor 1 is  $\dot{u}_1(0) = 0.4$  (this particular value is not significant, since one can rescale the time), and the initial velocity of impactor 2 is zero,  $\dot{u}_2(0) = 0$ . One can observe that for a value of coupling  $\varepsilon = 0.058$  the energy remains localized at impactor 1. A minimal increase of the coupling to  $\varepsilon = 0.059$  yields a qualitative change of the behavior: the impactors exchange energy. This process can be identified as nonlinear beating in the vibro-impact system.

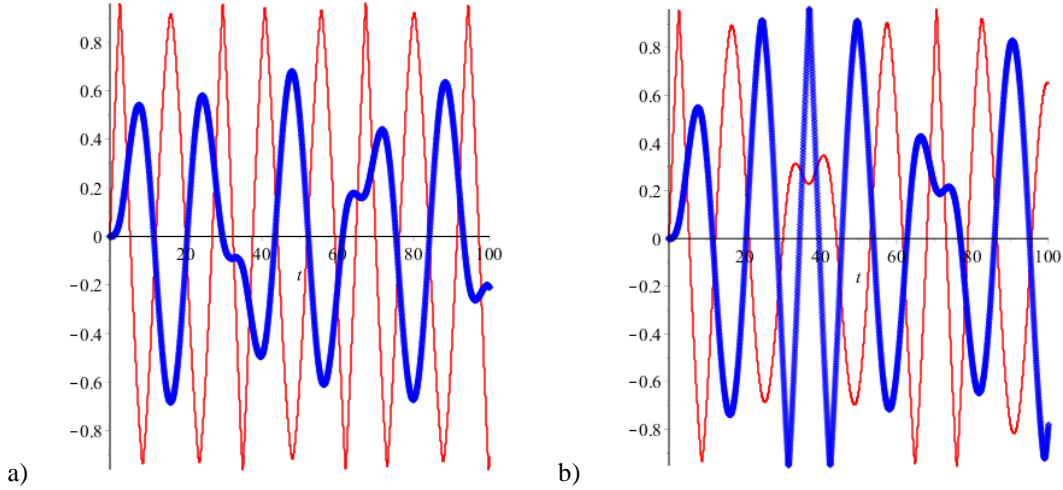


Figure 2. Time series for displacements in coupled vibro-impact system (Figure 1) with nonzero initial velocity of one impactor; a)  $\varepsilon = 0.058$ , b)  $\varepsilon = 0.059$ . Red (thin solid) line -  $u_1(t)$ , blue (thick points) line -  $u_2(t)$ .

In terms of the AA variables, Hamiltonian of the system presented in Figure 1 is written as follows:

$$H = \frac{\pi^2(I_1^2 + I_2^2)}{8} + \frac{2\varepsilon}{\pi^2} (\arcsin(\sin \theta_1) - \arcsin(\sin \theta_2))^2 \quad (7)$$

Averaging according to (4-6) yields the following set of the phase portraits on  $(\vartheta, \gamma)$  surface:

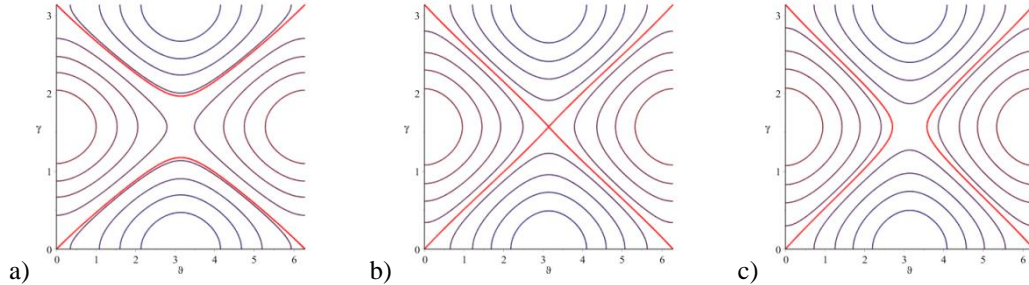


Figure 3. Phase portraits of the averaged system with Hamiltonian (32) for a)  $\varepsilon = 0.58$ ; b)  $\varepsilon = 0.6$ ; c)  $\varepsilon = 0.62$ . Thick red line denotes the limiting phase trajectory (LPT).

We obtain that the transition to the energy exchange takes place for  $\varepsilon_{cr} = 0.06$  in excellent agreement with numeric results presented in Figure 2

## Conclusions

The findings presented above lead to the conclusion that the averaging based on the action-angle variables offers a convenient framework for exploration of structure and bifurcations of the slow flow, including transitions from the localization to the energy exchange. The averaging procedure may be justified *a posteriori*, due to slowing down the dynamics due to passage of the phase trajectory of interest close to the saddle point. Thus, the averaging procedure may be justified even in the absence of the formal small parameter.

## References

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