Capturing similarity solutions in multidimensional Burgers' equation

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Summary. The topic of this talk are similarity solutions in multi-dimensional Burgers' equation.
We use the symmetries of the d-dimensional Burgers' equation to derive an equivalent partial differential algebraic equation (PDAE). In this formulation similarity solutions become true steady states.
Then we introduce a new and easily implementable numerical scheme for the PDAE, based on an IMEX-Runge-Kutta approach for a spatial semi-discretization. The time discretization is second order consistent and this order is also observed numerically.
Furthermore, our method enables us to obtain good approximations of similarity solutions by direct forward simulations and we are also able to observe meta-stable behavior near N-wave-like patterns.

Symmetries, similarity solutions and freezing for multi-dimensional Burgers' equation

The subject of this talk are similarity solutions in the multi-d Burgers' equation, which we write as an evolution equation in the form

$$u_t = \nu\Delta u - \tfrac{1}{p}\operatorname{div}\big(r|u|^p\big) =: F(u),\ u(x,t)\in\mathbb{R}, \tag{1}$$

where $r\in\mathbb{R}^d\setminus\{0\}$ is some vector, $p>1$ is some number and $\nu>0$ stands for the amount of viscosity present in the equation. The equation is a standard test equation for numerical schemes for hyperbolic conservation laws ($\nu=0$) and for hyperbolic dominated problems ($0<\nu\ll 1$).
A simple observation for (1) is, that the vectorfield F satisfies the following symmetry property: For $\alpha>0$, $b\in\mathbb{R}^d$ and a function $u\in H^2(\mathbb{R}^d)$ let $A(\alpha,b)$ denote the transformation $[A(\alpha,b)u](x)=\frac{1}{\alpha}u\big(\frac{x-b}{\alpha^{p-1}}\big)$. Then holds for $u\in H^2(\mathbb{R}^d)$

$$F\circ A(\alpha,b)(u) = \frac{1}{\alpha^{2p-2}}A(\alpha,b)\circ F(u) \quad \text{as equality in } L^2.$$

The key idea now is to make the *ansatz* that the solution u of (1) is of the form

$$u(x,t) = \tfrac{1}{\alpha(\tau(t))}v\Big(\frac{x-b(\tau(t))}{\alpha(\tau(t))^{p-1}}\Big). \tag{2}$$

The following theorem relates the differential equations solved by u and by v and is the basis for the numerical method we introduce below.

Theorem. *Assume that* $u_0\in H^2_s := \{u\in H^2(\mathbb{R}^d) : \operatorname{div}(xu)\in L^2(\mathbb{R}^d)\}$ *and* $\mu_1\in\mathcal{C}([0,\widehat{T});\mathbb{R})$, $\mu_2\in\mathcal{C}([0,\widehat{T});\mathbb{R}^d)$ *for some* $\widehat{T}>0$. *Let* $\alpha\in\mathcal{C}^1([0,\widehat{T});\mathbb{R}_+)$, $b\in\mathcal{C}^1([0,\widehat{T});\mathbb{R}^d)$, *and* $\tau\in\mathcal{C}^1([0,T);[0,\widehat{T}))$ *satisfy*

$$\alpha'(\tau)=\alpha(\tau)\mu_1(\tau),\ \alpha(0)=1,\quad b'(\tau)=\alpha(\tau)^{p-1}\mu_2(\tau),\ b(0)=0,\quad \dot{\tau}=\alpha(\tau)^{2-2p},\ \tau(0)=0. \tag{3}$$

Then $u\in\mathcal{C}^1([0,T);L^2)\cap\mathcal{C}([0,T);H^2_s)$ *solves the Cauchy problem for* (1) *with* $u(x,0)=u_0(x)$ *if and only if* $v\in\mathcal{C}^1([0,T);L^2)\cap\mathcal{C}([0,T);H^2_s)$, *given by* $u(t)=A\big(\alpha(\tau(t)),b(\tau(t))\big)v(\tau(t))$, *solves*

$$v_\tau = \nu\Delta_\xi v - \tfrac{1}{p}\operatorname{div}_\xi\big(r|v|^p\big) - \big[(1-p)\operatorname{div}_\xi(\xi v) + \big(pd-d-1\big)v\big]\mu_1 + \nabla v^\top\mu_2,\ v(\xi,0)=u_0(\xi). \tag{4}$$

Motivated by the theorem, we call a function u a similarity solution of (1) if it is a solution and is given as

$$u(t) = A\big(\alpha(\tau(t)),b(\tau(t))\big)v_*,$$

where v_* is a fixed element from H^2_s. In fact, a similarity solution is always given by a fixed element v_* from H^2_s and α,b,τ satisfying (3) with constant $\mu_1=\mu_{1*}\in\mathbb{R}$ and $\mu_2=\mu_{2*}\in\mathbb{R}^d$. It is not difficult to see that this restricts the parameter p from (1) to the value $\frac{d+1}{d}$.

The continuous freezing system

Note that in the above theorem there is a certain ambiguity. Namely, any solution v,μ_1,μ_2 of (4) leads to the same solution u of (1) by defining α,b,τ by (3). In particular, if (1) is a well-posed PDE, the PDE (4) is not well-posed because of the $d+1$ degrees of freedom due to μ_1,μ_2. The idea of the freezing method, originally introduced in [2], is to add $d+1$ algebraic equations to this problem and obtain a so-called partial differential algebraic equation. In the current case this leads to the PDAE system

$$\partial_\tau v = \nu\Delta_\xi v - \tfrac{1}{p}\operatorname{div}_\xi\big(r|v|^p\big) - \big[(1-p)\operatorname{div}_\xi(\xi v) + \big(pd-d-1\big)v\big]\mu_1 + \nabla v^\top\mu_2,\ v(\xi,0)=u_0(\xi), \tag{5a}$$

$$0 = \Psi(v,\mu_1,\mu_2), \tag{5b}$$

$$\frac{d}{d\tau}\alpha = \mu_1\alpha,\ \alpha(0)=1,\quad \frac{d}{d\tau}b = \alpha^{p-1}\mu_2,\ b(0)=0, \tag{5c}$$

$$\frac{d}{dt}\tau = \alpha(\tau)^{2-2p},\ \tau(0)=0, \tag{5d}$$

which has to be solved numerically. Following [2], we call (5b) the phase conditions. Moreover, we note that (5c), (5d) decouple and could be solved in a post processing step.

Numerical discretization

To solve the PDAE system (5a), (5b) is a challenging numerical task. In particular, we are interested in a numerical scheme which is suitable for all positive values of the viscosity constant ν. Note that (5a) consists of heterogeneous hyperbolic parts and of parabolic parts. For small ν the hyperbolic part dominates and for large values of ν the parabolic part dominates.

Spatial semi-discretization
We adapt a finite volume scheme for hyperbolic conservation laws and convection-diffusion equations from [4] to the case of heterogeneous convection-diffusion equations. The important features of the scheme are that it is suitable also in the purely hyperbolic case and that it is a central scheme, which does not require a precise knowledge of the solution structure of the related Riemann problems. This spatial discretization of (5a), (5b) leads to a huge method-of-lines DAE of the form

$$\begin{aligned} V' &= -H^0(V) - H^1(V)\mu + P(V), \\ 0 &= \Psi^h(V, \mu). \end{aligned} \tag{6}$$

IMEX-Runge-Kutta discretization
Because of the hyperbolic-parabolic coupled structure of the PDE-part (5a), the differential equation part of (6) consists of very different terms. On the one hand, the terms $-H^0(V) - H^1(V)\mu$ are highly nonlinear and very expensive to be solved in an implicit scheme but they also lead to a moderate CFL condition, so that one should solve the differential equation $V' = -H^0(V) - H^1(V)\mu$ with an explicit time-marching scheme. On the other hand the term $P(V)$ comes from the discretization of the Laplace-operator and hence the differential equation $V' = P(V)$ is very stiff for fine discretizations, so that one should rather use an implicit scheme. To couple these contrary requirements we adapt the idea of IMEX-Runge-Kutta schemes, cf. [1], to differential algebraic equations.
More precisely, we couple an explicit Runge-Kutta scheme and a diagonal implicit Runge-Kutta scheme to solve the method-of lines DAE (6). The algebraic constraint is solved in a half-explicit fashion, see [3]. Then we have the following result concerning the consistency error:

Theorem. *The IMEX-Runge-Kutta scheme which couples the explicit method of Heun with the implicit Crank-Nicolson scheme is*

- ***second order** consistent in V and μ at smooth solutions of* (6)*, if* (6) *is a DAE of **differentiation index 1**,*
- ***second order** consistent in V at smooth solutions of* (6)*, if* (6) *is a DAE of **differentiation index 2**.*

Numerical experiments and conclusions

For the results of numerical experiments we refer to [5] and [6]. It can be seen that the method indeed is of second order. Moreover, it is capable of approximating similarity solutions by a direct forward simulation in the sense that the solution to the Cauchy problem for the PDAE converges to a steady state $(v_*, \mu_{1*}, \mu_{2*})$ of the PDAE, which yields a similarity solution, as explained above. Calculating a good approximation of a similarity solution (or relative equilibrium) is always a crucial and very important first step, needed for a subsequent bifurcation analysis or stability analysis of the patterns.
Finally, our method works perfectly well for all ranges of ν and thus the scheme should be suitable also for other problems of coupled hyperbolic-parabolic structure and as such may become an important tool for the analysis of patterns in hyperbolic-parabolic coupled problems in general.
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