Evolutionary Dynamics of Membership Distribution Functions of a Forced Triple Well Potential System with Fuzzy Uncertainty

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Summary. Transient responses of a forced triple-well potential system with fuzzy uncertainty are studied by means of the Fuzzy Generalized Cell Mapping (FGCM) Method. The FGCM method is first introduced. A rigorous mathematical foundation of the FGCM is established as a discrete representation of the fuzzy master equation for the possibility transition of continuous fuzzy processes. The FGCM offers a very effective approach for solutions to the fuzzy master equation based on the min-max operator of fuzzy logic. A fuzzy response is characterized by its topology in the state space and its possibility measure of membership distribution functions (MDFs). This paper focuses on the evolution of transient MDFs of the fuzzy response. It is found that as the time goes on, transient MDFs spread around three potential wells. The evolutionary orientation of transient MDFs aligns with unstable invariant manifolds leading to stable invariant sets. An example with multiplicative fuzzy noise is given.

Fuzzy Generalized Cell Mapping (FGCM)

Consider a dynamical system with fuzzy uncertainty,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, S), \quad x \in \mathbf{D}, \tag{1}$$

where $\mathbf{x}$ is the state vector, t the time variable, S a fuzzy set with a membership function $\mu_S(s) \in (0,1]$ where $s \in S$, and $\mathbf{f}$ is a vector-valued nonlinear function of its arguments. $\mathbf{D}$ is a bounded domain of interest in the state space. When the system parameter S is a fuzzy number, Equation (1) is a fuzzy differential equation.
The equation of the FGCM system is given as follows by discretizing the time t, state variables $\mathbf{x}$ and the fuzzy set S in Equation (1)

$$\mathbf{p}(n+1) = \mathbf{P} \circ \mathbf{p}(n), \;\; \mathbf{p}(n) = \mathbf{P}^n \circ \mathbf{p}(0), \qquad p_i(n+1) = \max_j \; \min[p_{ij}, p_j(n)], \tag{2}$$

where $\mathbf{P}^{n+1} = \mathbf{P} \circ \mathbf{P}^n$ and $\mathbf{P}^0 = \mathbf{I}$. $\circ$ denotes the min-max operation. The matrix $\mathbf{P}$ denotes the one-step transition possibility matrix, $\mathbf{P}^n$ denotes the n-step transition possibility matrix. The vector $\mathbf{p}(n)$ is called the n-step membership distribution vector and $\mathbf{p}(0)$ the initial membership distribution vector. The (i, j)th element p_{ij} of the matrix $\mathbf{P}$ is called the one-step transition possibility from cell j to cell i. Equation (2) describes the evolution of a fuzzy response process with its MDFs.
Consider the fuzzy master equation for the possibility transition of continuous fuzzy process [1],

$$p(\mathbf{x}, t) = \sup_{\mathbf{x}_0 \in \mathbf{D}} [\min\{ p(\mathbf{x}, t | \mathbf{x}_0, t_0),\; p(\mathbf{x}_0, t_0)\}], \quad \mathbf{x} \in \mathbf{D} \tag{3}$$

where $\mathbf{x}$ is a fuzzy process, $p(\mathbf{x}, t)$ is the membership distribution function of $\mathbf{x}$ at t, and $p(\mathrm{x}, t, \mathbf{x}_0, t_0)$ is the transition possibility function. Equation (2) of the FGCM can be viewed as a discrete representation of fuzzy master equation (3). A partial differential equation from Equation (3) for continuous time processes has been derived by Friedman and Sandler [1]. This equation is analogous to the Fokker-Planck-Kolmogorov equation for the probability density function of stochastic processes. The solution to this equation is in general very difficult to obtain analytically. The FGCM offers a very effective method for solutions to this equation, particularly, for fuzzy nonlinear dynamical systems [2].

Evolutionary Dynamics of Membership Distribution Functions (MDFs)

Consider a forced Duffing oscillator with a triple-well potential driven by multiplicative fuzzy noise,

$$\begin{cases} \dot{x} = y \\ \dot{y} = -0.35y - x + 0.5x^3 - 0.05x^5 + S\cos t \end{cases}, \tag{4}$$

where S is a fuzzy parameter with a triangular MDF,

$$\mu_S(s) = \begin{cases} [s - (s_0 - \varepsilon)]/\varepsilon, \; s_0 - \varepsilon \le s < s_0 \\ -[s - (s_0 + \varepsilon)]/\varepsilon, \; s_0 \le s < s_0 + \varepsilon \\ 0, \quad otherwise \end{cases}. \tag{5}$$

where $\varepsilon > 0$ is a parameter characterizing the intensity of fuzziness of S and is called fuzzy noise intensity. s_0 is the nominal value of S with membership grade $\mu_S(s_0) = 1$. We take s_0=0.2, ε =0.7.

The domain $\mathbf{D}$=$\{-3.5 \le x \le 3.5; -2.0 \le y \le 2.0\}$ is discretized into 141×141 cells. The 5×5 sampling points are used within each cell. S is discretized into 401 segments. Hence, out of each cell, there are 10,025 trajectories with varying membership grades. Transient MDFs are shown in Figures 1 and 2. T is one step mapping step.

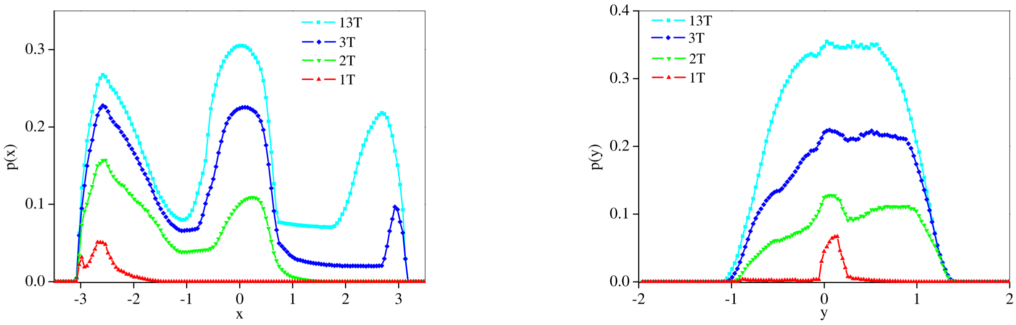

Fig. 1. Evolutionary process of the marginal MDFs of displacement x and velocity y for the fuzzy Duffing equation (4) with the intensity of fuzzy noise ε =0.7 at different times 1T, 2T, 3T and 13T.

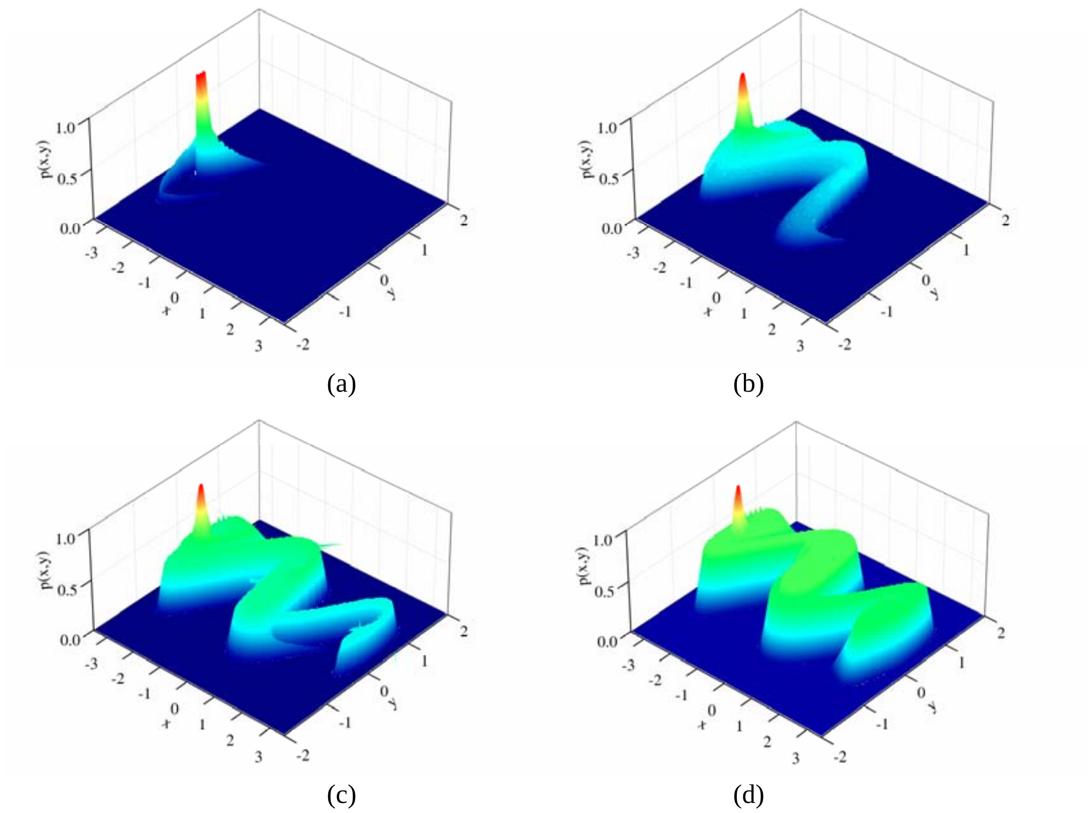

Fig. 2. Evolutionary process of the joint MDFs of displacement x and velocity y for the fuzzy Duffing equation (4) with the intensity of fuzzy noise ε =0.7 at different times 1T, 2T, 3T and 13T

Concluding Remarks

This paper investigates the evolution of transient MDFs of the fuzzy response. It is found that the evolutionary orientation of transient MDFs aligns with unstable invariant manifolds leading to stable invariant sets. As the time goes on, the mean values of transient MDFs increase moving towards to form three possibility peaks around three potential wells. The results obtained herein are of value to real engineering problems. Especially, the evolution of transient state MDFs may be difficult to obtain with other conventional methods.

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References
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