Stochastic Asymptotic-preserving Galerkin methods for multiscale kinetic equations with uncertainties

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Summary. We introduce stochastic Galerkin (SG) methods for multiscale kinetic equations with uncertainties, and show that they can be made asymptotic-preserving, in the sense that the SG schemes preserve various asymptotic limits in the discrete space. These methods can capture the multiscale physics and uncertainty without numerically resolving the small scales. Rigorous analysis will be provided to prove the desired asymptotic properties. Examples to be discussed include the linear neutron transport equation and the nonlinear Boltzmann equation.

Introduction

Many kinetic and hyperbolic equations often have diffusive scaling that asymptotically leads to diffusion equations. When the equations are near the diffusive regime, numerical simulations become prohibitively expensive, due to the exceedingly small parameters, e.g., the particle mean free path, relaxation time, etc., that need to be numerically resolved. Asymptotic-preserving (AP) schemes are those that mimic the asymptotic transitions from the kinetic or hyperbolic equations to their diffusive limits in discrete setting. The AP strategy has been proved to be a powerful and robust technique to address multiscale problems in many kinetic and hyperbolic problems. The main advantage of AP schemes is that they are very efficient in the diffusive regime, for they do not need to resolve the small diffusive parameters numerically and yet can still capture the macroscopic behavior governed by the limiting diffusion equations. For example, it was proved, in the case of linear transport with a diffusive scaling, the AP scheme converges uniformly with respect to the scaling parameter . This is expected to be true for all AP schemes , although specific proofs are needed for specific problems. For many multi-scale problems, carefully constructed AP schemes avoid the difficulty of coupling a microscopic solver with a macroscopic one, as the micro solver automatically becomes a macro solver in the diffusive regime.

Formulations

In practical applications, kinetic and hyperbolic problems almost always contain parameters that are uncertain, due to modeling and/or experimental errors. In this paper, we investigate numerical methods for such problems and aim to develop stochastic version of the AP schemes that can quantify the uncertainty in the diffusive regime. While well established for the deterministic kinetic or hyperbolic equations, AP schemes have not been constructed for hyperbolic and kinetic problems with random inputs. In this paper, we present a set of stochastic AP methods for kinetic and hyperbolic problems with random inputs. To cope with the random inputs, we employ generalized polynomial chaos (gPC) approach , combined with stochastic Galerkin (SG) method.

Results and Discussions

While the gPC-SG approach has been adopted for a large variety of stochastic problems, this paper represents the first attempt, to our best knowledge, to construct stochastic AP (sAP) schemes. The sAP methods are constructed by extending the idea from the deterministic AP method,, and are illustrated via three prototype kinetic equations: the linear Goldstein-Taylor model, the nonlinear Carleman model, and the linear transport equation. The selected equations are representative, as they possess features such as linearity vs. nonlinearity, representations in physical space vs. in phase space. We will demonstrate that the carefully constructed gPC-Galerkin method can be AP, in the sense that in the diffusive regime it automatically becomes a gPC-Galerkin approximation for the limiting stochastic diffusion equations. This holds true for fixed time step, mesh size and order of the gPC expansions, and without the requirement for refinement.

Conclusions

It is interesting to note that one obvious approach to extend the deterministic AP methods to random domain is to adopt stochastic collocation (SC) approach. SC methods are based on sampling and can be easily applied to any systems with established deterministic solvers. One can apply a deterministic AP scheme to each individual samples, solve them separately to obtain the solution ensemble, and then construct an approximation, e.g., a gPC approximation, to the stochastic solution in the randoms space. However, it is important to note that even though each samples satisfy the AP properties, there is no guarantee that the global approximation is AP in the entire random space. This is especially true in higher dimensional random spaces, as the errors incurred by the construction (e.g., interpolation errors) can be highly non-trivial away from the samples. On the other hand, the proposed sAP is based on the Galerkin formulation, and can be proven to be AP throughout the random domain, for any fixed (and even low) order of gPC expansions.

References

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