Towards a bifurcation theory for random dynamical systems

Speaker. Jeroen S.W. Lamb, Imperial College London, London SW7 2Z, UK

Abstract. Despite the obvious relevance to applications, a bifurcation theory for random dynamical systems is still only in it's infancy. We discuss some low-dimensional examples that illustrate recent advances and challenges.

Introduction. The study of nonlinear dynamical systems has been one of the success stories of mathematics and science in the 20th century. It started with the pioneering work of Poincaré in the late 19th century and experienced strong acceleration since the 1970s, when theory started to be accompanied by numerical explorations on computers. The resulting insights have been instrumental to understand the behaviour of "complex systems" and also drove home the important fact that complicated behaviour and unpredictability can be intrinsic properties of dynamical systems with only a few degrees of freedom.

More recently, due to the wide availability of powerful computers, applied scientists have been taking "noise" into account in their models. But the theoretical treatment of noise in dynamical systems lags well behind current practise in applied modelling and simulations, resulting in an unwelcome gap between theory and practise. The field of random dynamical systems addresses the question of how the dynamical behaviour of nonlinear dynamical systems, driven by a random input signal, can be described and understood. Importantly, random dynamical systems naturally arise also in the context of deterministic dynamical systems that are not hyperbolic and are reducible to a skew-product form, where the base dynamics is essentially strongly chaotic and well-describable by a stochastic process.

Stochastic differential equations are special examples of random dynamical systems and many important developments have taken place to understand aspects of the nature of their solutions, with applications in many branches of the sciences. However, in comparison to the study of deterministic differential equations, in the stochastic context the theoretical understanding of the represented *dynamics* remains underdeveloped and is still in its infancy. For instance, while the existence of a stationary measure is an important property of a random dynamical system, it provides very little information about the underlying dynamics, which may be essential to address questions concerning (structural) stability and phase transitions or bifurcations in the presence of parameter variations.

The main difference between the state-of-the-art in deterministic dynamical systems and random dynamical systems is that in the deterministic setting there has been an important development (famously initiated by Poincaré) to not only treat differential equations from the point of view of analysis, but to add in topological and geometric characterizations that are complementary to the analytical one. The resulting qualitative theory of dynamical systems has provided powerful insights into the mechanisms behind observable dynamical behaviours which have enabled the scientific revolution of "chaos theory" to take hold in science.

A fundamental obstacle in the presence of *noise* is that many of the important technical concepts and natural objects, in terms of which we currently understand the qualitative behaviour of deterministic dynamical systems, cease to exist. Thus far the main contributions to the theory of random dynamical systems have come from directions where generalisations from the deterministic to the random setting have proven to be possible. A first challenge in the quest for a comprehensive theory of random dynamical systems is to connect these different strands of knowledge. Beyond that, it remains a major challenge to identify the key questions in the field. In this context, explorative case studies are fundamental to guide theoretical developments.

In the talk we briefly review the state-of-the-art in the develoment of a bifurcation theory for random dynamical systems with a focus on the following specific recent results:

- Uniformly versus non-uniformly attractive random equilibria: additive noise does not really destroy a pitchfork bifurcation [CDLR13,SDLR], cf [CF98].
- Shear-induced chaos: first analytical proof of existence of positive Lyaponov exponents for a stochastically driven limit cycle [ELR16], cf [LY08] and implications for the stochastic Hopf bifurcation.

CDLR13 Mark Callaway, Doan Thai Son, Jeroen S.W. Lamb and Martin Rasmussen. The dichotomy spectrum for random dynamical systems and pitchfork bifurcations with additive noise. arXiv:1310.6166 *to appear in Ann. Inst. H. Poincaré Probab. Statist.*

CF98 Hans Crauel and Franco Flandoli. Additive noise destroys a pitchfork bifurcation, Journal of Dynamics and Differential Equations **10**(2) (1998), 259–274.

ELR16 Maximilian Engel, Jeroen S. W. Lamb, and Martin Rasmussen, Bifurcation analysis of a stochastically driven limit cycle. arXiv:1606.01137

LY08 K. Lin and L.-S. Young, Shear-induced chaos, Nonlinearity **21** (2008), 899–922.

SDLR Yuzuru Sato, Doan Thai Son, Jeroen S.W. Lamb and Martin Rasmussen. Noise-induced chaos in a random logistic map. In preparation.