Numerical solving unsteady space-fractional problems

Petr Vabishchevich

Nuclear Safety Institute, Russian Academy of Sciences
52, B. Tulskaya, Moscow, Russia
e-mail: vabishchevich@gmail.com

Non-local applied mathematical models based on the use of fractional derivatives in time and space are actively discussed in the literature. Many models, which are used in applied physics, biology, hydrology and finance, involve both sub-diffusion (fractional in time) and super-diffusion (fractional in space) operators. Super-diffusion problems are treated as evolutionary problems with a fractional power of an elliptic operator. For example, suppose that in a bounded domain Ω on the set of functions $u(x) = 0, \ x \in \partial\Omega$, there is defined the operator $\mathcal{D}$: $\mathcal{D}u = -\triangle u, \ x \in \Omega$. We seek the solution of the Cauchy problem for a fractional power of an elliptic operator:

$$\frac{du}{dt} + \mathcal{D}^\alpha u = f(t), \quad 0 < t \leq T,$$

$$u(0) = u_0,$$

for a given $f(x,t)$, $u_0(x)$, $x \in \Omega$ using the notation $f(t) = f(\cdot, t)$.

We have proposed [1] a computational algorithm for solving an equation with fractional powers of elliptic operators on the basis of a transition to a pseudo-parabolic equation. For the auxiliary Cauchy problem, the standard two-level schemes are applied. The computational algorithm is simple for practical use, robust, and applicable to solving a wide class of problems. A small number of pseudo-time steps is required to reach a steady solution. This computational algorithm for solving equations with fractional powers of operators is promising when considering transient problems.

We construct unconditionally stable two- and three-level schemes for the approximate solution of unsteady problems with the fractional power of an elliptic operator. A transition to a new temporary level involves solving the standard elliptic problem for the desired solution. The main computational costs are associated with the evaluation of the right-hand side containing the fractional power of an elliptic operator.

An unsteady problem is considered for a space-fractional equation in a bounded domain. A first-order evolutionary equation involves the fractional power of an elliptic operator of second order. Finite element approximation in space is employed. To construct approximation in time, regularized two-level schemes are used. The numerical implementation is based on solving the equation with the fractional power of the elliptic operator using an auxiliary Cauchy problem for a pseudo-parabolic equation. The scheme of the second-order accuracy in time is based on a regularization of the three-level explicit Adams scheme. More general problems for the equation with convective terms are considered, too. The results of numerical experiments are presented for a model two-dimensional problem.

References

[1] P. N. Vabishchevich, Numerically solving an equation for fractional powers of elliptic operators, Journal of Computational Physics 282 (1) (2015) 289–302.