Analytical studies of a two degree-of-freedom vibro-impact system

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Summary. A system of two coupled oscillators, with two rigid barriers imposed on the second one, is considered. The method of multiple scales is used in combination with a saw-tooth function. The relation describing the slow invariant manifold is found. The final approximate solutions are of semi-analytical nature. Occurance of the two response regimes is verified: periodic motion and strongly modulated response. The interplay between the system parameters is analyzed.

Introduction

Most of approximate analytical methods applicable to nonlinear dynamical systems take their roots from the classical perturbation approach, and usually their usage is limited to weakly nonlinear problems [7]. Among practically important cases of strong nonlinearities, the vibro-impact systems are the ones for which general analytical solutions are just impossible. However, in recent years, several approaches have been developed by imposing certain conditions on motion of such systems, e.g. a combination of the multiple scales method with a saw-tooth function [1, 2, 3, 4], the non-smooth temporal transformation (NSTT) [5], the concept of impact modes [6].
The first abovementioned technique has been applied to systems with the vibro-impact nonlinear energy sink (VI NES), in the context of the targeted energy transfer (TET). Only impact interactions between the NES and the primary oscillator have been considered. In what follows, a more complicated NES configuration, involving elastic and viscous components, is studied.

Mathematical model

Consider a system of two coupled oscillators illustrated in Fig. 1. The stiffness constants of linear springs are denoted by k_1, k_2, while the damping coefficients are c_1, c_2. The bodies are interconnected by a purely nonlinear (cubic type) spring with a constant k_2'. It is assumed that mass of the second body is relatively small ($m_2 \ll m_1$). Moreover, motion of this oscillator is restricted by two rigid barriers; the restitution coefficient is denoted by κ. The system is subjected to the external excitation: $F_1(t) = F_{10}\sin(\omega_1 t)$.
Using the displacements x_1 and x_2 as the generalized coordinates, one can write the equations of motion of the system in the non-dimensional form:

$$\begin{aligned} &\ddot{X}_1 + X_1 + \gamma_1\dot{X}_1 - \beta_2(X_2 - X_1)^3 - \gamma_2(\dot{X}_2 - \dot{X}_1) = f_{10}\sin(\Omega_1\tau) \\ &\varepsilon\ddot{X}_2 + \Omega_{20}^2 X_2 + \beta_2(X_2 - X_1)^3 + \gamma_2(\dot{X}_2 - \dot{X}_1) = -\varepsilon(\kappa+1)\sum_j \dot{X}_2(\tau)\delta(\tau - \tau_j) \end{aligned} \tag{1}$$

where $X_1 = x_1/L$, $X_2 = x_2/L$, an overdot denotes differentiation with respect to the dimensionless time $\tau = \omega_{10}t$, the mass ratio $\varepsilon = m_2/m_1$ plays a role of the small parameter, and

$$\omega_{10}^2 = \frac{k_1}{m_1}, \quad \Omega_1 = \frac{\omega_1}{\omega_{10}}, \quad \Omega_{20}^2 = \frac{k_2}{k_1}, \quad \gamma_1 = \frac{c_1}{m_1\omega_{10}}, \quad \gamma_2 = \frac{c_2}{m_1\omega_{10}}, \quad \beta_2 = \frac{k_2' L^2}{m_1\omega_{10}^2}, \quad f_{10} = \frac{F_{10}}{m_1 L\omega_{10}^2}. \tag{2}$$

Moreover, $\delta(\bullet)$ is the Dirac delta function whereas τ_j is a time instance of the jth impact. Some of the quantities are assumed to be small, thus, the following parameters are formally introduced: $\gamma_1 = \varepsilon\hat{\gamma}_1$, $\gamma_2 = \varepsilon^2\hat{\gamma}_2$, $\beta_2 = \varepsilon^2\hat{\beta}_2$, $f_{10} = \varepsilon\hat{f}_{10}$, $\Omega_{20}^2 = \varepsilon\hat{\Omega}_{20}^2$.

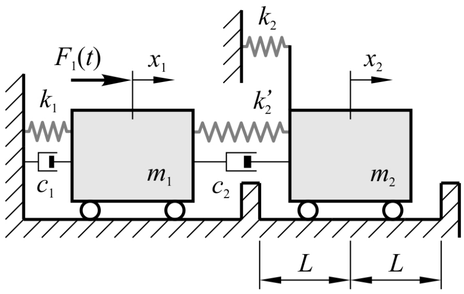

Figure 1: The vibro-impact system to be considered

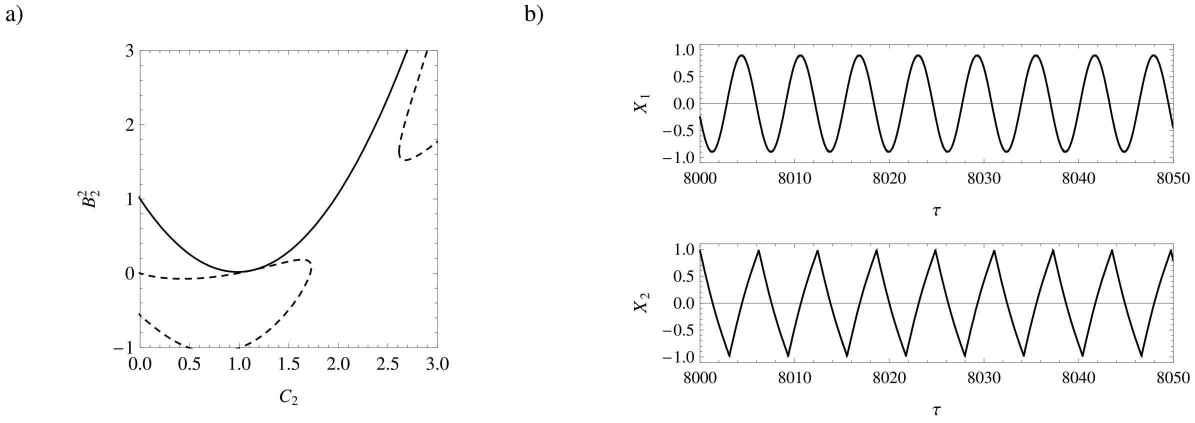

Figure 2: Semi-analytical results: a) the SIM (solid) and the curve $B_2^2 = g(C_2)$ (dashed), b) response of the system

Asymptotic analysis and results

Consider the simplest case of TET, i.e. 1:1 resonance with two impacts per cycle. Two detuning parameters are used: $\Omega_1 = 1 + \varepsilon\hat{\sigma}_1$, $\Omega_{20}^2 = 1 + \varepsilon\hat{\sigma}_2$. In order to analyze the system, two time scales ($\tau_k = \varepsilon^k\tau$ with $k = 0, 1$) are introduced: $X_i = X_{i0}(\tau_0, \tau_1) + \varepsilon X_{i1}(\tau_0, \tau_1)$. At the approximation of zero order we get

$$D_0^2 X_{10} + X_{10} = 0, \qquad D_0^2 X_{20} + X_{20} = -(\kappa + 1)\sum_j D_0 X_{20}\delta(\tau_0 - \tau_{0j}) \tag{3}$$

The solutions can be expressed in the form

$$X_{10} = B_1(\tau_1)\sin(\tau_0 + \phi_1(\tau_1)), \qquad X_{20} = B_2(\tau_1)\sin(\tau_0 + \phi_2(\tau_1)) + \frac{2}{\pi}C_2(\tau_1)\arcsin[\cos(\tau_0 - \theta_2(\tau_1))] \tag{4}$$

where the second part of X_{20} is the saw-tooth function that describes impacts at $\tau_0 = j\pi + \theta_2$ for $j = 0, 1, 2, \ldots$. Analysis of the impact conditions leads to the relation between B_2 and C_2 which defines the slow invariant manifold (SIM):

$$C_2 = \frac{1 \pm \sqrt{1+\rho^2}\sqrt{B_2^2 - B_{2\,\mathrm{min}}^2}}{1+\rho^2}, \qquad B_{2\,\mathrm{min}} = \frac{\rho}{1+\rho^2}, \qquad \rho = \frac{2(1-\kappa)}{\pi(1+\kappa)} \tag{5}$$

At the next level of approximation, the solvability conditions provide an additional relation: $B_2^2 = g(C_2)$. The fixed points of the slow-flow are found graphically as the intersection of the new curve and the SIM (5). In Figure 2a, an example is presented for the set of parameters: $\kappa = 0.65$, $\hat{\gamma}_1 = \hat{\gamma}_2 = 0.1$, $\hat{\beta}_2 = 0.1$, $\hat{\sigma}_1 = \hat{\sigma}_2 = 0.1$, $\hat{f}_{10} = 0.2$. Due to complexity of the differential equations for B_2 and ϕ_2, they are solved numerically. Hence, the final steady-state solutions have a semi-analytical character. For the given case, the system response is shown in Fig. 2b for $\varepsilon = 0.1$.
Investigations of the semi-analytical solutions are focused mainly on stability of the fixed points located on the two branches of the SIM. As indicated in literature (e.g. see [3, 4]), the vibro-impact systems can exhibit not only periodic motion, but also strongly modulated response (SMR). Occurance of such behaviour in the studied case is verified. The analytical results are compared to purely numerical solutions of Eqs. (1). The interplay between particular parameters is tested, especially the balance between the excitation and the two forms of dissipation is analyzed.

Conclusions

Semi-analytical solutions for a vibro-impact system with elastic and viscous components are obtained. Thanks to the assumption of small mass ratio, the method of multiple scales can be used. However, this approach does not require the restitution coefficient to be close to unity. In further works, the presented procedure may be extended to more complex cases, including one-sided impacts.

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