Dynamics of a Small Stiff Spherical Particle in an Acoustic Standing Wave in Fluid

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Summary. The movement of a small stiff spherical particle in the standing wave pressure field in fluid is studied taking into consideration the Basset force. The integro-differential equation is solved using the Krylov-Bogolyubov averaging technique and the efficient numerical scheme for the solution of the obtained averaged integro-differential equation is proposed. The transition from "fast" variables to "slow" averaged coordinates provided the averaged particle trajectory and its qualitative behavior. It is shown that the Basset force may play serious role in focusing small particles by ultrasound and even may change their focusing position from standing wave nodes to antinodes. The obtained results are investigated analytically and numerically.

Introduction

The problem of particle dynamics in an acoustic wave was firstly investigated in [1]. The averaged forces acting on a particle were attributed as radiation forces which appearance was caused by acoustic wave scattering on a particle and this approach was widely used in many subsequent works. The results presented in these works included the particle compressibility effect and describe well the particle motion in the inviscid fluid. This approach is valid for small viscosity fluids and rather big particles (compared with viscosity penetration depth during an acoustic wave oscillation period).

As the most promising applications of the acoustophoresis are biological ones(see for many examples [2]) the size of manipulated objects may be of the order of microns or smaller. The viscous effects may play significant role in this case. The radiation force approach is improved and extended to practically all viscosity coefficient values in the ingenious work [3] by using the Prandtl-Schlichting boundary-layer theory.

In this work the approach of local forces acting on a particle is used. The particle is considered to be subjected to gravitational, inertial and viscous forces and the interaction force between the particle and the acoustic wave is calculated as the force acting on particle in the non-stationary fluid flow around the particle. The Maxey-Riley equation [4] is used and the gravitational and Saffman forces are neglected because of their smallness compared to the other forces. It is shown that the Basset force which corresponds to the nonstationarity of the flow may play significant role in the particle motion and even change its focusing position from standing wave nodes to antinodes.

Particle movement equations

The Basset-Boussinesq-Oseen equation (BBO equation) is used to describe the motion of a small spherical particle in unsteady flow with velocity field $v(x,t)$ at low Reynolds numbers

$$\begin{gathered}
(m_{\text{particle}} + m_{\text{added}})\ddot{x} = F_{\text{inertial}} + F_{\text{Stokes}} + F_{\text{Basset}} + F_{\text{Faxen}} + F_{\text{g}},\\
m_{\text{particle}} = \tfrac{4}{3}\pi\rho_p a^3, \quad m_{\text{added}} = \tfrac{2}{3}\pi\rho a^3,\\
F_{\text{inertial}} = \tfrac{4}{3}\pi\rho a^3\left(w + \tfrac{W}{2}\right), \quad w = \tfrac{\partial v}{\partial t} + v\tfrac{\partial v}{\partial x}, \quad W = \tfrac{\partial v}{\partial t} + \dot{x}\tfrac{\partial v}{\partial x},\\
F_{\text{Stokes}} = -6\pi\mu a(\dot{x} - v),\\
F_{\text{Basset}} = -6\pi\mu a t_\mu^{1/2}\left(\tfrac{d}{dt}\right)^{1/2}(\dot{x} - v), \quad t_\mu = \tfrac{\rho a^2}{\mu},\\
\left(\tfrac{d}{dt}\right)^{1/2}(\dot{x} - v) = \int_{t'=0}^{t}\left(\tfrac{d^2x}{dt'^2} - W(t')\right)\tfrac{dt'}{\sqrt{\pi(t-t')}} + \tfrac{\dot{x}(0) - v(0)}{\sqrt{\pi t}},\\
F_{\text{Faxen}} \sim \mu a^3\nabla^2 v \sim \mu a v(a/\lambda)^2,\\
F_{\text{g}} = \tfrac{4}{3}\pi a^3(\rho_p - \rho)g \ .
\end{gathered} \tag{1}$$

Here ρ_p and ρ stand for particle and fluid densities, a stands for particle radius, μ stands for fluid viscosity which we consider not to depend on density. By w and W are denoted fluid acceleration along fluid and along the particle trajectory. The Basset force is used in the form proposed by [5] which contains the corrections for nonzero initial particle relative velocity with the fluid. The inertial force is used in the Maxey-Riley form [4].

The last two forces are neglected because of their smallness and (1) is divided by $(2/3)\pi a^3$

$$\begin{gathered}
(\rho + 2\rho_p)\ddot{x} = 2\rho w + \rho W - \tfrac{9\mu}{a^2}\left(\tfrac{dx}{dt} - v\right) + F_{\text{B}},\\
F_{\text{B}} = -\tfrac{9\sqrt{\rho\mu}}{a}\left(\int_{t'=0}^{t}\left(\tfrac{d^2x}{dt'^2} - W(t')\right)\tfrac{dt'}{\sqrt{\pi(t-t')}} + \tfrac{\dot{x}(0) - v(0)}{\sqrt{\pi t}}\right) \ .
\end{gathered} \tag{2}$$

The dimensionless acoustic wave amplitude is denoted by $b \ll 1$. It is defined as the ratio of maximal fluid velocity in the wave to the sound speed in the fluid. The dimensionless variables $q = (\omega/c)x$, $\tau = \omega t$ and the dimensionless viscosity parameter $K = \frac{3\mu}{\omega\rho a^2 b} \sim 1$ are introduced, ω stands for the wave angular frequency. The asymptotic averaged equation for $b \to 0$ is obtained using the Krylov-Bogoliubov averaging technique [6] as described in [7]. The solution comprises the following steps:

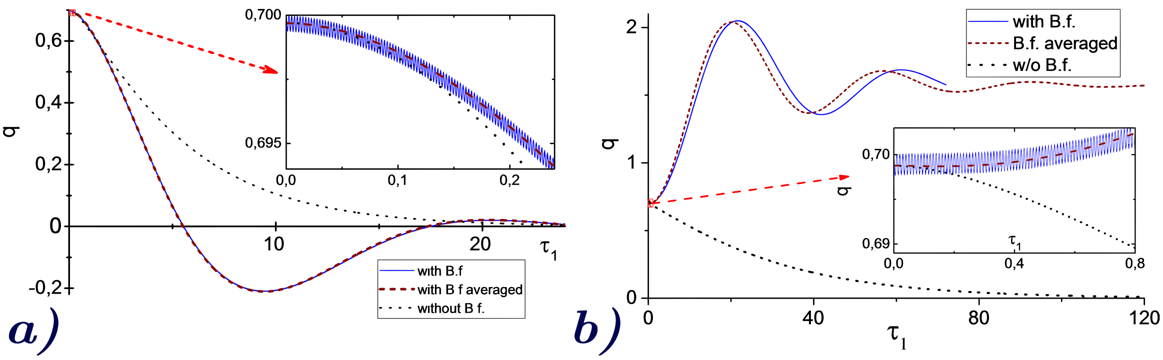

Figure 1: The asymptotic and the exact numerical solutions of the integro-differential BBO equation (light solid line and dashed line) and the numerical solution of the same equation without the Basset force (dotted line). The inset shows the magnified part of the dependence. $\tau_1 = 3\rho b\tau/(2\rho_p + \rho)$ stands for the "slow" dimensionless time. Part a) illustrates the heavy particle behavior and b) illustrates a light particle for which the Basset force consideration changes the particle focus point. On both parts $K = 3$ and $h = 0.05$.

1. The fast motion of particle in standing wave is marked out by a certain variable change.
2. The system of equations is transformed to standard form for the Krylov-Bogolyubov averaging technique application.
3. The system averaging procedure is applied and the new variables depending only on "slow" time ($\tau_1 = 3\rho b\tau/(2\rho_p+\rho)$) are introduced.
4. The obtained averaged integro-differential equation is solved numerically and compared with the exact numerical solution of the initial system (2).

The initial and the averaged equations were modeled using a 5-point-grid numerical scheme for the Basset integral calculation of the third order of precision. As the initial conditions are not defined very well a step variation technique was used in order to make much smaller step at the beginning of the time grid.

Results and conclusion

In the Figure 1.a) the trajectory of heavy particle $\rho_p/\rho = 1.5$ is modeled for $b = 16/3 \cdot 10^{-4}$. The trajectory for small b practically coincides with the asymptotic solution at $b \to 0$. The initial equation simulation took about eight hours of one core CPU time and the averaged integro-differential equation simulation took only 170 milliseconds using the same numerical scheme. The ability to compare the results for these two approaches up to such a big time ($\tau_{\max} = 60000$, $1.2M$ steps) shows the advantage of our scheme over the one used in [7] where only the small part of the particle trajectory was calculated.

In the Figure 1.b) is illustrated one of the most interesting results of the Basset force inclusion. Here the Basset force changes the particle equilibrium point from pressure nodes to antinodes. The trajectory of light particle $\rho_p/\rho = 0.6$ is modeled for $b \approx 1.17 \cdot 10^{-3}$. The averaged solution describes rather well the solution of the initial equation although b value may be a little bit too high value for the exact coincidence. The calculations of the initial equation here took about six hours compared with 50 ms for the averaged equation solution.

Considering the obtained results one may note that the cavitation threshold for atmospheric pressure is around $b \approx 4.4 \cdot 10^{-5}$ and the typical small parameter b values are much less than used. One may conclude that the obtained asymptotics will describe the particle motion even better at smaller b and the proposed averaging procedure will be the only possibility to obtain the results in a reasonable computation time (as the time increases proportional to b^{-2}).

The proposed technique was used for modeling particle behavior at different particle densities and fluid viscosities. The focusing point changes at $\rho_p/rho = 1$. The proposed theory is applicable for the small particles described by the low Reynolds numbers such as the biological particles in blood subjected to the low intensity ultrasound. The proposed results may be useful in the acoustophoresis method in medicine.

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