Rigidity constraints in Analytical Dynamics

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Summary: In recent years, Udwadia et al. [1] have proposed to obtain dynamical equations using Lagrange method with generalised parameters as quaternions q. In 2014, a different point of view was applied by the actual author to treat problems whatever the nature of the parameters. Since rigidity is not (a priori) included, the main aim is the necessary use of stress tensor in the Virtual Work Principle (VWP), then its elimination for rigid bodies. Here we propose to show the applicability of our method to friction and division.

Background.

If body forces are not present for simplicity, the VWP is written for a body B

$$-\int_B \rho a.v\,dx + \int_\Gamma \varphi.v\,da - \int_B \sigma : grad v\,dx = 0$$

where ρ is the density, a the acceleration, φ the surface forces, σ the stress tensor, and v the virtual piecewise displacements. In the application of some rotational motion, $x=R(q(t))X$, x being the actual position of the particle X, the virtual displacements are $v=(R'_i R^{-1}x)w_i$ where the w_i's are arbitrary and R is a 3x3 matrix function of quaternions. R'_I is the partial derivative of $R(q_1,\ldots,q_n)$. R is not necessarily a rotation, i.e. the constraint $q^Tq=1$ is not fulfilled as an a priori condition.
If we take account of the actual virtual displacements in the above formula,, then the first term is the virtual work

(denoted L_iw_i) of acceleration. Then we have

$$grad\ v= (R'_i R^{-1})w_i=S_iw_i+A_iw_i,\quad \sigma : grad v = (\sigma : S_i) w_i$$

where S_i and A_i are resp. the symmetrical and anti-symmetrical parts of the matrix $R'_i R^{-1}$. Now in order to eliminate the stress tensor, we require the relations $S_iw_i=0$.(sum on i), a priori realised if R is a rotation. In addition, it is seen that surface forces f occur by global quantities only (i.e. $R(f)$ and $M(f)$). So the following compatibility conditions result: whatever the w_i's such that $S_iw_i=0$, we have

$$[-L_i+M(f)\, a_i]w_i=0 \quad \text{(sum on } i\text{)}$$

(a_i: dual vector of matrix A_i) under the only above hypotheses. Finally we write the rigidity constraint $\boldsymbol{q^Tq=1}$ when quaternions are used..

Example 1: Contact with friction.

We consider an homogeneous rigid wheel (centre O, radius r and mass m) rolling in a vertical plane $O_0x_0y_0$ on an inclined line (or surface) O_0X_0 under the gravitational acceleration g downwards, the gravitational force being ($f=-mgy_0$) applied on the centre O of the wheel. We use the referential $Ref=O_0X_0Y_0Z_0$ with the angle between O_0x_0 and O_0X_0 noted a . Two-dimensional Euler parameters (p,q) are introduced to specify the rotation of the wheel, so

$$R_{11}=R_{22}=1-2q^2\ ,\ R_{12}=-R_{21}=-2pq\ ,\ R^{-1}=R^T/\Delta\quad ,\quad \Delta=1+4q^2(p^2+q^2-1$$

Now we introduce the virtual coefficients (w_x,w_y,w_p,w_q) associate to the parameters (x,y,p,q) and the condition $w_iS_i=0$, i.e. $pw_p+qw_q=0$. Under the above condition, the VWP is writing

$$-\int_B \rho a.v\,dx - mgy_0 .v(O)+Tv_1(A)+Nv_2(A)=0$$

where $(T,N,0)$ are the components of the two-dimensional contact force on the wheel applied at the contact point A. Now we must use the contact law of friction, by example in the hypothesis of a bilateral contact $(y=r)$ at the point $A=(x,y-r,0)$ of the wheel, implying the geometric constraint $y=r$, together with the Coulomb law of friction equivalent to the inequality of Duvaut and Lions

$$T\,[v_1(A)-u_1(A)]+k|N|[|v_1(A)|-|u_1(A)|]\geq 0$$

First the parameters are specified such that $w_x=w_p=w_q=0$, satisfying $w_iS_i=0$. It results $v(x)=(0,w_y,0)$ so that by taking account of the bilateral contact $y=r$

$$mgcosa-N=0 \text{ and } \dot{K} + mgsina\ \dot{x} + k|N||u_1(A)| =0$$

$$\int_B \rho a.v\,dx + mgy_0\, v(O)-Nv_2(A)+ k|N||v_1(A)| \geq 0 \quad \text{where } N=mg\ cosa$$

that is available whatever the parameters (w_x, w_p, w_q). After some straightforward calculus, the acceleration term is obtained under the form

$$\int_B \rho a.v\,dx = m\ddot{x}w_x + a_{11}\ddot{p} + a_{22}\ddot{q} + 2a_{12}\dot{p}\dot{q} + 2b\dot{q}^2$$

$$a_{11} = 2mr^2(q^2w_p + pqw_q),\ a_{22} = 2mr^2(pq + p^2 + 4q^2)w_q,\ a_{12} = 2mr^2(qw_p + pw_q),\ b = 4mr^2qw_q$$

Taking account of this expression, the differential variational inequality follows

$$(m\ddot{x} + mg\sin a)w_x + kmg\cos a\left|w_x + r(\alpha_p w_p + \alpha_q w_q)\right| + 2mr^2(Aw_p + Bw_q) \geq 0$$

(where α_p, α_q and A,B are given functions) under the compatibility condition $pw_p+qw_q=0$. That is the basic relation to solve the problem completed naturally by initial conditions on velocities (and positions).

Example 2: Dividing a rigid body.

We consider a rigid body B^* divided (on a virtual manner) into two parts B_1 and B_2. We note B the system of these two bodies and S their common boundary. Each part may be viewed as a continuum and the precedent theory may be applied to the system B described by parameters $q=(q_o,q_1,\ldots,q_n)^T$. In fact we introduced the respective motions

$$x^{(a)}=T^{(a)}+R^{(a)}X^{(a)} \quad a=1,2$$

where T and R are functions of parameters q.and then the associate virtual displacement
Naturally, the constraints of the type $w_iS_i=0$ must be fulfilled on each part in order to eliminate the Cauchy stress tensor in the interior of each of the two bodies B_1 and B_2. But we must write geometric constraints of continuity on the common boundary, viz $x^{(1)}=x^{(2)}$ on the surface S, i.e.

$$\Delta\ T=T^{(2)}-T^{(1)}=0 \ \text{ and }\ \Delta\ R=R^{(2)}-R^{(1)}=0$$

This join is realised by local Cauchy forces along the common boundary S, but, in our hypothesis of displacements, resultants and moments only may be introduced to take account of these forces. So, in the Virtual Work Principle applied to B, we must introduce

$$J = \int_S \varphi^{(1)}.v^{(1)}\ da + \int_S \varphi^{(2)}.v_2^{(2)}\,da$$

where resultant R and moment M are defined on surfaceS. But this quantity is related to Cauchy stress tensor and in our framework must be eliminated since they are interior forces of the entire body B^*. This condition is satisfied if we choose the virtual displacements such that

$$w_i\ \Delta\frac{\partial T}{\partial q_i} = 0 \quad , \quad w_i\ \Delta\alpha(A_i) = 0$$

As an example we consider a system of two cylindrical masses B' (mass m') and B_3 (mass m_3) moving without friction along the x-axis. They are connecting by a linear-spring (k_3) and the mass B' is connected to the origin by a linear-spring (k_0). We decomposed the mass B' into two sub-bodies B_1 and B_2 of masses m_1 and m_2. So we are considering a multi-body system. Parameters describing the system are the respective coordinates q_1,q_2 and (q_2+q_3)

$$T_1=(q_1,0,0),\ T_2=(q_2,0,0),\ T_3=(q_2+q_3,0,0)$$

Rigidity constraint of B' is then $\Delta\ T=T_2-T_1=0$, i.e. $q_2=q_1$. So the virtual principle is reduced to

$$\{-L_i+[R(f)+R(\varphi)].T'_i\}w_i=0$$

$$[R(f).T'_i]w_i=[-k_0q_1]w_1+[k_3q_3]w_2+[-k_3q_3](w_2+w_3)$$

Since $\Delta T = T_2-T_1=(q_2-q_1)(1,0,0)$, we have the constraint $(w_2-w_1=0)$ and for any (w_1,w_3)

$$(-m_1\ddot{q}_1 - k_0q_1)w_1 + (-m_2\ddot{q}_2 + k_3q_3)w_2 + (-m_3(\ddot{q}_2+\ddot{q}_3) - k_3q_3)w_3 = 0$$

and naturally the geometric constraint of rigidity $q_2=q_1$. These equations may be obtained by other methods.

Conclusion.

The present work has presented a natural link existing between Analytical Dynamics and Continuum Mechanics. The key of our scheme was the use of the Virtual Work Principal. Then the elimination of Cauchy stresses introduces compatibility relations between virtual coefficients.

References.

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