Proper orthogonal decomposition of delay-differential equations

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Summary. In this paper we present an approach for investigating the stability of time-delay differential equations. Our method is based on the fact that delay differential equations can be defined as initial-boundary value problems. We use Proper Orthogonal Decomposition by stability analysis and study the influence of different choices of parameters on distribution of singular values.

Introduction

Stability analysis of time-delay equations is a still-growing field. We take the Hayes equation as a prototype model:

$$\dot{x}(t) = ax(t) + bx(t-\tau), \tag{1}$$
$$x(t) = \theta(t), \quad -\tau \le t \le 0. \tag{2}$$

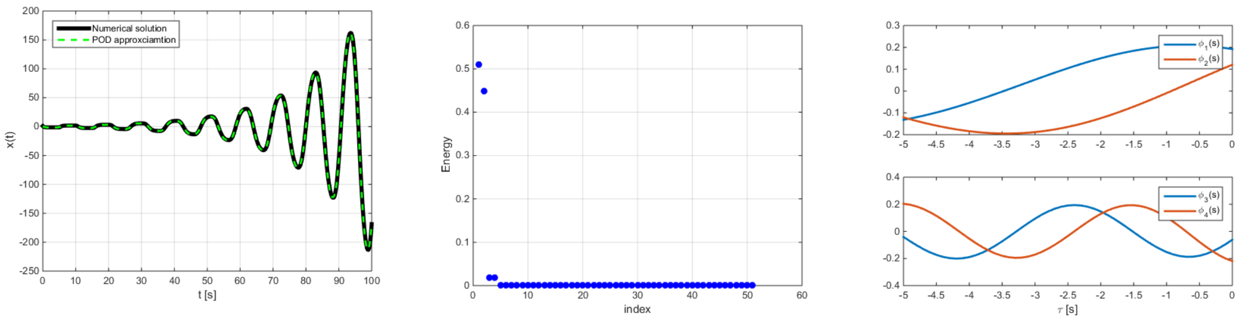
By introducing the so called shift of time $u(t,s) = x(t+s), \quad s \in [-\tau, 0)$ the initial value problem (Eqs. (**??**), (**??**)) can be recast into the following partial differential equation [**?**]:

$$\frac{\partial u(t,s)}{\partial t} = \frac{\partial u(t,s)}{\partial s}, \quad s \in [-\tau, 0], \tag{3}$$
$$\left.\frac{\partial u(t,s)}{\partial t}\right|_{s=0} = au(t,0) + bu(-\tau,t), \tag{4}$$
$$u(t,s) = \theta(s), \quad s \in [-\tau, 0]. \tag{5}$$

The two-dimensional function $u(t,s)$ can be viewed as a spatio-temporal representation of the "process"(the initial value problem Eqs. (**??**),(**??**)), where the shift of time s can be interpreted as the spatial dimension. We numerically integrated Eq. (**??**), and transformed the solution into a matrix. We used Singular Value Decomposition to factorizing the data matrix, which gave us three matrices. Two matrices contain the singular vectors and a diagonal matrix contains the singular values. The SVD is an analogous process to the principal axis transform in higher dimension. The singular values role is simlar to principal moments of inertia and singular vectors like the principal axes. We explore the connection between the singular values and the stability of the equation.

Results

A vizsgalatok soran arra jutottunk, hogy a POD segitsegvel kapott bazis fugvenyekkel jol kozelutheto a dde megoldas. Az egyik esetet mutatja be az abara **??**



(a) Numerikus megoldas, kozelito megoldas POD bazisfÃijggvenyekkel

(b) Singular values

(c) The first four POD basis functions

Figure 1: A Hayes egyenlet megoldasa $a = -3$ es $b = -4$ mellett, es a megoldashoz tartozo szingularisertekek es bazis fuggvenyek

The stability chart for a and b parameters can constructed in an analytical way Hayes [**?**][**?**]. To examine the connection between singular values and the stability of the equation, we move along a section between a stable and an instable point on the a, b plane. This lines are ilustrated in Fig. **??**
A **??** kepen abrazoltuk azokat az eseteket, amikor fix a ertekek mellett b valtoztatatva vizsgaltuk a szingularis ertekeket.

Az abrakon piros szaggatott vonallakkal jeloltuk b parameter azon erteket, ahol az egyenlet elveszíti a stabilitasat. Elvarasainknak megfeleloan jelentos valtozas tapasztalhato a stabilitasi hatarokon valo atlepesnel. Vizsgalataink alapjan az is kijelentheto, hogy fix negativ a ertekek melett a b valtozasaira hasonloan reagal a viszgalat egyenlet.

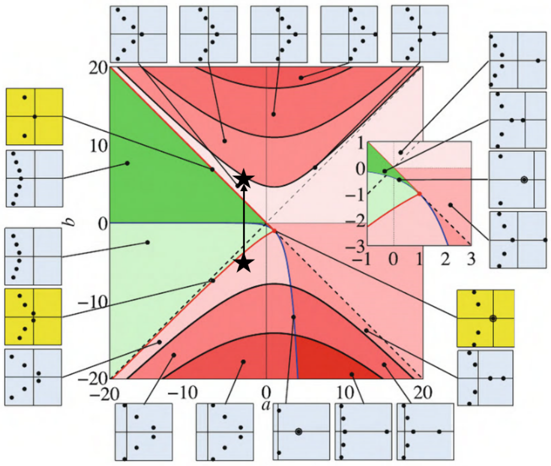
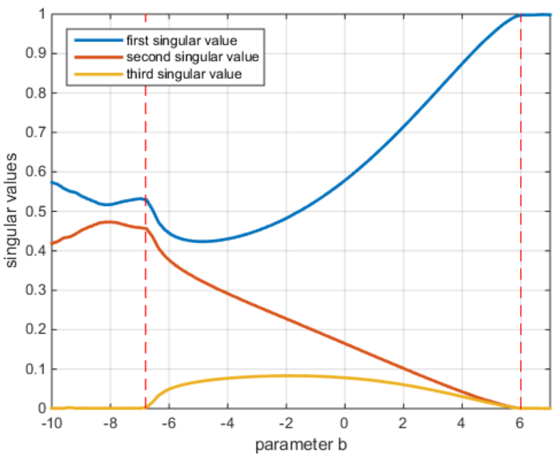
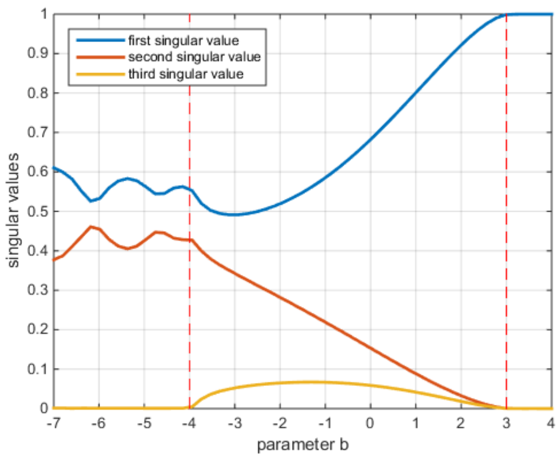


Figure 2: Stability chart of Hayes equation



(a) $a = -3$ and $b \in [-7, 4]$



(b) $a = -6$ and $b \in [-10, 7]$

Figure 3: Az elso harom szingularis ertek valtozasa a b paramater valtoztatasaval

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