A Chaotic Linear Operator on the Space of Odd $2\pi-$Periodic Functions

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Summary. Not just nonlinear systems, but infinite-dimensional linear systems can exhibit complex behavior. It has long been known that twice the backward shift on the space of square-summable sequences l_2 displays chaotic dynamics. Here we construct the corresponding operator $\mathcal{C}$ on the space of 2π-periodic odd functions and provide its representation involving a principal value integral. We explicitly calculate the eigenfunction of this operator, as well as its periodic points. We also provide examples of chaotic and unbounded trajectories of $\mathcal{C}$.

Introduction

Linear systems has commonly been thought to exhibit relatively simple behavior. Surprisingly, infinite dimensional linear systems can have complex dynamics. In particular, Rolewicz in 1969 [1] showed that the backward shift B multiplied by 2 (i.e. $2B$) on the space of square summable sequences l_2 exhibits transitivity (and thus gives rise to chaotic dynamics). Here we construct a chaotic linear operator $\mathcal{C}$ by "lifting" $2B$ to the space L_2 of square-integrable functions (more precisely to the Hilbert space $L_2(0,\pi)$ of 2π-periodic odd functions). We state and prove a theorem about expressing this shift on $L_2(0,\pi)$ in terms of a Principal Value integral. Then we define and analyze the corresponding chaotic operator $\mathcal{C}$ on $L_2(0,\pi)$, including finding its eigenvectors and periodic points. We provide examples of unbounded and chaotic trajectories of $\mathcal{C}$.

A Chaotic Linear Operator on the Space of 2π-periodic Odd Functions

The backward shift B on the infinite-dimensional Hilbert space l_2 of square-summable sequence is defined as

$$Ba = (a_2, a_3, \ldots), \tag{1}$$

where $a = (a_1, a_2, \ldots)$, such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. The Hilbert space $L_2(0,\pi)$ of square-integrable functions is isomorphic with l_2 (by the Riesz-Fischer theorem) and is a natural functional representation of the sequence space l_2. By odd extension, elements of $L_2(0,\pi)$ can be viewed as odd 2π-periodic square-integrable functions so that $L_2(0,\pi)$ is also isomorphic with the space of odd 2π-periodic square-integrable functions. Now we "lift" $a \in l_2$ to $L_2(0,\pi)$ by the summation

$$f(t) = \sum_{n=1}^{\infty} a_n \sin nt. \tag{2}$$

We define the backward shift $\mathcal{B}$ acting on $L_2(0,\pi)$ as

$$\mathcal{B}f(t) = \sum_{n=1}^{\infty} a_{n+1} \sin nt = \sum_{n=1}^{\infty} a_n \sin(n-1)t. \tag{3}$$

Our main result is that

Theorem 1 *$\mathcal{B}f(t)$ can be expressed as*

$$\mathcal{B}f(t) = f(t)\cos t - \frac{1}{\pi}\mathrm{PV}\int_0^{\pi} \frac{\sin t \sin\xi}{\cos t - \cos\xi} f(\xi) d\xi. \tag{4}$$

Our "chaotic" operator (twice the backward shift) is now defined as

$$\mathcal{C}f(t) = 2\mathcal{B}f(t) = 2f(t)\cos t - \frac{2}{\pi}\mathrm{PV}\int_0^{\pi} \frac{\sin t \sin\xi}{\cos t - \cos\xi} f(\xi) d\xi. \tag{5}$$

Analysis of $\mathcal{C}$

We explicitly obtained the eigenfunctions of $\mathcal{C}$, its fixed point and periodic orbits [2]. We can also create a function that gives rise to a chaotic orbit under the action of $\mathcal{C}$. First, we note that for $2B$ (on l_2) the point

$$y = \left(\frac{y_1}{1}, \frac{y_2}{2}, \frac{y_3}{4} \ldots\right), \tag{6}$$

where y_i is the i-th digit of a normal irrational number (whose digits are uniformly distributed), generates a chaotic orbit. π is believed to be normal, so we take y_i to be the i-th digit of π. We now lift this point to $L_2(0,\pi)$ using Equation (2):

$$\Psi(t) = \sum_{i=1}^{\infty} \frac{y_i}{2^{i-1}} \sin nt. \tag{7}$$

Figure 1 shows the first 10 elements of the orbit of Ψ under the action of $\mathcal{C}$, i.e. $\mathrm{Orb}\,(\mathcal{C}, \Psi)$. The first element of the orbit is Ψ itself.

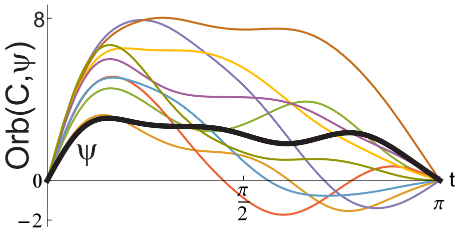

Figure 1: The first 10 elements of $\mathrm{Orb}\,(\mathcal{C}, \Psi)$.

Figure 2 shows the orbit $\mathrm{Orb}\,(\mathcal{C}, \Psi)$ evaluated at three different t's ($\pi/20, \pi/2, 19\pi/20$) for 200 iteration.

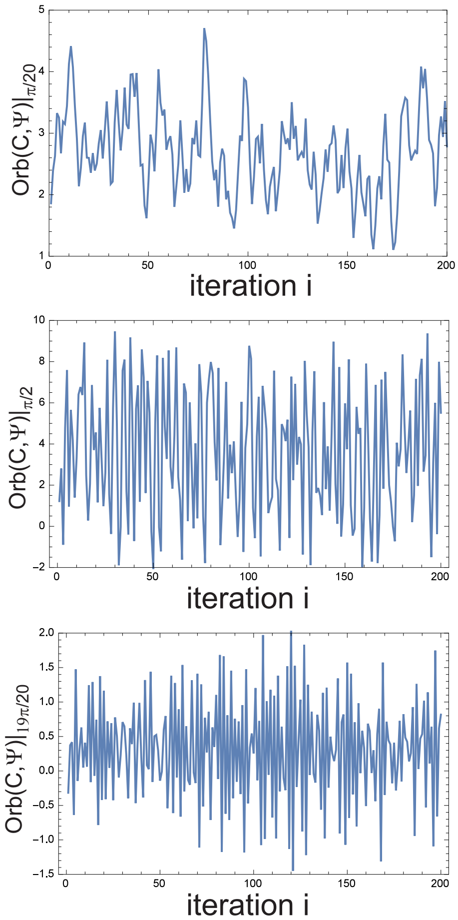

Figure 2: The orbit $\mathrm{Orb}\,(\mathcal{C}, \Psi)$ for 200 iterations evaluated at $t = \pi/20, \pi/2, 19\pi/20$.

Conclusions

Contrary to common belief, linear systems can display complicated dynamics. Starting from twice the backward shift on l_2 we constructed the corresponding shift operator C on $L_2\,(0, \pi)$ (the space of odd, $2\pi-$periodic functions) and provided its representation using a modicum of distribution theory and Cauchy's principal value integral. We explicitly calculated the periodic points of the operator (including its nontrivial fixed point) and provided examples of chaotic and unbounded trajectories of $\mathcal{C}$.

References

[1] Rolewicz S. (1969) On orbits of elements. Studia Mathematica **32**(1):17-22.

[2] Kalmár-Nagy T. and Kiss M. (2017) Complexity in Linear Systems: A Chaotic Linear Operator on the Space of Odd $2\pi-$Periodic Functions. *Complexity*