Stability and Vibration Amplitude of the Quasi Periodic Delayed Mathieu Equation with Frequency-Modulated Coefficients

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Summary. In this work the periodic coefficient of the delayed Mathieu equation is modified similarly to the Frequency Modulation. The quasi-stationary solution of the governing equation is derived and the maximal amplitude is approximated by L_1 norm. The stability map is also generated based on the Hill-Flouqet theory. The results are validated by numerical time integration.

Frequency-Modulated Delayed Mathieu Equation with External Forcing

The main motivation for this work is the mathematical modelling of the Sinusoidal Spindle Speed Variation (SSSV) in milling [1, 2]. The milling processes are usually described by differential equations with time-periodic coefficients and point delay. In case of SSSV the time periodic coefficients are modulated. It is shown, that this method can efficiently be used to suppress the self-excited vibration (so-called chatter), however, the stationary vibration is usually not modelled in the literature, but it can be an important factor while it defines the surface properties [7]. The simplest analogue model of the milling process which can already capture the emerging phenomenon is the Delayed Mathieu Equation (DME) [3]. In this paper the model is extended by a Frequency Modulated (FM) periodic coefficients and external sinusoidal forcing.

$$\ddot{x}(t) + \kappa\dot{x}(t) + \big(d + \varepsilon\cos(\phi(t))\big)x(t) - bx(t-\tau) = F_0\cos(\omega_F t), \tag{1}$$

$$\phi(t) = 1t + \beta\sin(\omega_m t), \tag{2}$$

$$\dot{\phi}(t) = 1 + \beta\omega_m\cos(\omega_m t). \tag{3}$$

Note, that the periodicity of the time-delay is neglected in this model and a fixed $\tau = 2\pi$ is used, because in some cases the variable transport delay can be eliminated (see [4]). The coefficient of $x(t)$ can be considered as a single-tone modulated wideband FM signal for which the instantaneous frequency is defined by (3) with the tone frequency ω_m, carrier frequency 1 and modulation index (tone amplitude) β. Based on the following identity

$$e^{i\beta\sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta)e^{in\omega_m t}, \tag{4}$$

where J_n is the Bessel function of the first type of order n. It is clear that $\omega_{k,l} := k + l\omega_m$ can modulate the forcing signal, thus the following trial function is used:

$$x(t) = \big(a_{k,l}e^{i\omega_{k,l}t}\big)e^{i\omega_F t} = a_{k,l}e^{i(\omega_{k,l}+\omega_F)t}. \tag{5}$$

Note, that we assume ω_m to be irrational, which leads to a quasi-periodic solution described by independent $a_{k,l}$ parameters. Substituting (5) into (1) and applying (4), we get the following formula:

$$\sum_{q,p=-\infty}^{\infty}\sum_{k,l=-\infty}^{\infty} e^{i(\omega_{q,p}+\omega_F)}\Big(\big(\delta_{q-k}\delta_{p-l}(i(\omega_{k,l}+\omega_F))^2\big) + \big(\kappa\delta_{q-k}\delta_{p-l}(i(\omega_{k,l}+\omega_F))\big) +$$

$$\big(d\delta_{q-k}\delta_{p-l} + \frac{\varepsilon}{2}J_{p-l}(\beta)(\delta_{q-k-1}+\delta_{q-k+1})\big)\Big)a_{k,l} = \sum_{q,p=-\infty}^{\infty} e^{i(\omega_{q,p}+\omega_F)}F_0\delta_{q-k}\delta_{p-l},$$

where δ is the Kronecker delta function. Applying the harmonic balance for each $\omega_{q,p} + \omega_F$ component [5, 6], one can end up with a following formula

$$A^H_{q,p,k,l}a_{k,l} = F_{q,p}, \tag{6}$$

where $A^H_{q,p,k,l}$ is an infinite block Hill matrix in which all the elements are infinite Hill matrices themselves. Truncating the indices of this fourth order tensor, a finite approximation of the $x(t)$ can be generated. Note, that the stability boundaries can also be found in the parameter space if the determinant of the block matrix is zeros for any ω_F (see: [5]).
In some application only the maximal value of the solution $x(t)$ is necessary [7], which can be approximated by L_1 norm of the resultant coefficients:

$$\max_t |x(t)| \approx \|a_{k,l}\|_{L_1} = \sum_{k,l}|a_{k,l}|. \tag{7}$$

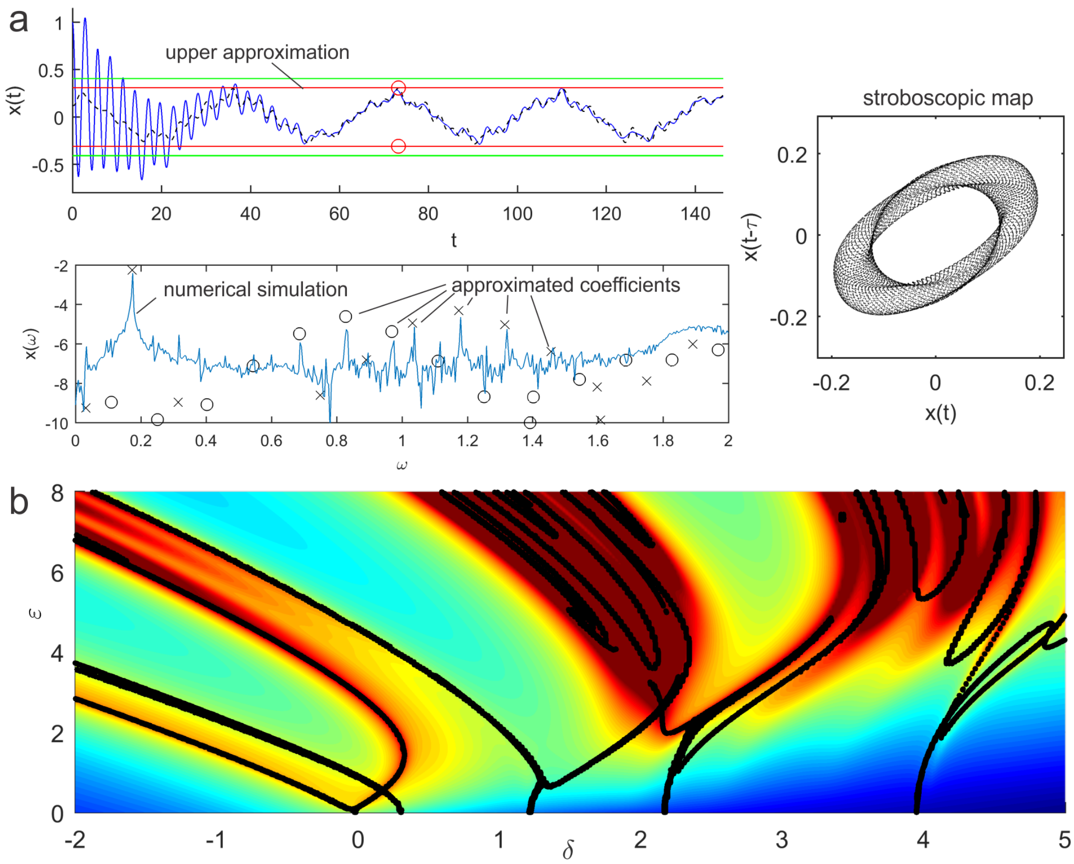

Figure 1: a) time signal of a numerical simulation of the frequency modulated delayed Mathieu equation with quasi-periodic excitation, its stroboscopic map and the corresponding spectrum b) Colormap denotes the maximal position, the black lines denote the corresponding stability boundary

Numerical results

The proposed algorithm is tested and compared with numerical time-domain simulations. The time signal in Fig.1a shows that after a transient phase the numerical simulation tends to the approximated solution. The corresponding spectrum contains all the components as assumed in Eq.(5) and some others belong to the transient part. The upper estimation in Eq.(7) provides a close approximation for the maximal value of the simulated quasi-periodic oscillations. The approximated maximal values in the function of δ and ε are plotted in Fig.1b by shaded color. Figure 1a presents the stability boundary by black lines where the determinant is zero.

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