Linear Flows in the Rapid Distortion Limit: Dynamical Systems Analysis of the Kelvin-Townsend Equations

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Summary. The Kelvin-Townsend equations are the Fourier space analogue of the Navier-Stokes equations in the Rapid Distortion Limit. The bifurcation occurring at the case of homogeneous shear flow is classified and explained. The oscillatory behavior and stability transition for elliptic flows is analyzed via Floquet theory.

Introduction

Even though turbulence is completely described by the Navier-Stokes equations, for most complex turbulent flows of engineering interest, the direct numerical solution of these equations is not yet computationally feasible. Hence, studies of turbulent flows usually incorporate some level of modeling, e.g. Reynolds stress models, subgrid scale models, two-point closures, etc. The concept of Reynolds averaging leads to the Reynolds-averaged Navier-Stokes equations, and this statistical approach necessitates the introduction of closure models. The specific terms that need modeling include the dissipation, the transport and the pressure-strain correlation terms. It is widely accepted that pressure-strain correlation plays a crucial role in the evolution and structure of turbulent flows. In this work, we are interested in the evolution of simple flowfields in the rapid distortion limit, i.e. when the mean flow timescale is much smaller than that of the fluctuating flow. In this case the nonlinear interactions among fluctuating modes can be neglected and therefore the evolution equations -the so-called rapid distortion equations- are linear in fluctuating velocity. The Fourier space equivalent of the rapid distortion equations, the Kelvin-Townsend equations, represents a fundamental dynamical system in fluid mechanics. By analyzing the Kelvin-Townsend equations from the dynamical systems perspective we point out the changing action of pressure in different mean flow classes (hyperbolic, shear and elliptic). Insight into the role of pressure is expected to contribute to the understanding of the hyperbolic and elliptic instabilities [1].

The Kelvin-Townsend equations

The Kelvin-Townsend equations

$$\begin{aligned}
\tfrac{de_1}{dt} &= -e_1 - ce_2 + e_1(e_1^2 - e_2^2),\\
\tfrac{de_2}{dt} &= ce_1 + e_2 + e_2(e_1^2 - e_2^2),\\
\tfrac{de_3}{dt} &= e_3(e_1^2 - e_2^2),\\
\tfrac{du_1}{dt} &= (2e_1^2 + 2ce_1e_2 - 1)u_1 - (2ce_1^2 + 2e_1e_2 - c)u_2,\\
\tfrac{du_2}{dt} &= (2ce_2^2 + 2e_1e_2 - c)u_1 - (2e_2^2 + 2ce_1e_2 - 1)u_2,\\
\tfrac{du_3}{dt} &= 2e_3(e_1 + ce_2)u_1 - 2e_3(e_2 + ce_1)u_2.
\end{aligned} \tag{1}$$

govern the evolution of the components of the Fourier transforms of the fluctuating velocity field in the rapid distortion limit. Here (u_1, u_2, u_3) are the Fourier velocity coefficients and (e_1, e_2, e_3) is the unit wavenumber vector. The "ellipticity" parameter $c = \sqrt{\frac{\beta}{1-\beta}}$ describes the nature of the linear flow.

The dynamics of the wavenumber vector

The first three equations of (1) govern the evolution of the unit wavenumber vector. The e_1, e_2 evolution equations are decoupled from the e_3 equation (we also have the constraint $e_1^2 + e_2^2 + e_3^2 = 1$), therefore we can restrict our attention to the $e_1 - e_2$ plane.

$$\begin{aligned}
\tfrac{de_1}{dt} &= -e_1 - ce_2 + e_1(e_1^2 - e_2^2),\\
\tfrac{de_2}{dt} &= ce_1 + e_2 + e_2(e_1^2 - e_2^2),
\end{aligned} \tag{2}$$

The origin $\mathrm{P}_0 = [0, 0]$ is always a fixed point. For $\beta < 0.5$ (the regime of hyperbolic flows) there are four other fixed points on the unit circle $e_1^2 + e_2^2 = 1$ given by

$$\begin{aligned}
P_1 = [-\bar{e}_1, \bar{e}_2],\ P_2 = [\bar{e}_1, -\bar{e}_2],\\
P_3 = [\bar{e}_2, -\bar{e}_1],\ P_4 = [-\bar{e}_2, \bar{e}_1],
\end{aligned} \tag{3}$$

where

$$\bar{e}_1 = \sqrt{\frac{1 + \sqrt{\frac{2\beta-1}{\beta-1}}}{2}}, \quad \bar{e}_2 = \sqrt{\frac{1 - \sqrt{\frac{2\beta-1}{\beta-1}}}{2}}. \tag{4}$$

In this regime P_0 is a saddle point, P_1 and P_2 are unstable nodes, while and P_3 and P_4 are stable nodes. For $\beta > 0.5$ it is possible to explicitly express the $e_1 - e_2$ solutions. Introducing the transform $e_1 = x + y, e_2 = x - y$ equation 2 reduces to the form

$$\begin{aligned} x' &= \left(4x^2 + c - 1\right) y, \\ y' &= \left(4y^2 - c - 1\right) x. \end{aligned} \tag{5}$$

which yields (with initial conditions $(x_0, 0)$)

$$x = \frac{x_0\sqrt{c-1}\cos(\Omega t)}{\sqrt{c - 1 + 2x_0^2\left(1 - \cos(2\Omega t)\right)}}, \quad y = \frac{x_0\sqrt{c+1}\sin(\Omega t)}{\sqrt{c - 1 + 2x_0^2\left(1 - \cos(2\Omega t)\right)}}, \quad \Omega = \sqrt{c^2 - 1}. \tag{6}$$

This system has the first integral

$$x^2 + \frac{c - 1 + 4x_0^2}{c+1} y^2 = x_0^2, \tag{7}$$

which corresponds to ellipses in the $x - y$ plane. Reverting back to the solutions for the variables in unit wavenumber space one gets

$$e_1 = \frac{x_0(\sqrt{c-1}\cos(\Omega t) + \sqrt{c+1}\sin(\Omega t))}{\sqrt{c - 1 + 2x_0^2\left(1 - \cos(2\Omega t)\right)}}, \quad e_2 = \frac{x_0(\sqrt{c-1}\cos(\Omega t) - \sqrt{c+1}\sin(\Omega t))}{\sqrt{c - 1 + 2x_0^2\left(1 - \cos(2\Omega t)\right)}}. \tag{8}$$

The dynamics of the velocity vector

For the regime of elliptic flows, that is $\beta \in (0.5, 1]$, the solutions of e_i are periodic functions of time. Thus, the evolution is governed by an ordinary differential equation with periodic coefficients. The evolution of the u_1 and u_2 components is independent of the u_3 component, therefore we can focus on the u_1-u_2 dynamics. It is known that any second order homogeneous ordinary differential equation with periodic coefficients can be reduced to the form of the Hill's equation [2]. We introduce the parameters

$$\begin{aligned} \gamma &= (c+1)\, x + (1-c)\, y, \\ \delta &= (c+1)\, x + (c-1)\, y. \end{aligned} \tag{9}$$

The trace and determinant of the coefficient matrix are

$$\begin{aligned} &\tau = 2e_1\gamma - 2e_2\delta = 8xy, \\ &\Delta = (-1 + 2e_1\gamma)(1 - 2e_2\delta) - (-c + 2e_2\gamma)(c - 2e_1\delta) = \left(1 - c^2\right)\left(4x^2 + 4y^2 - 1\right). \end{aligned} \tag{10}$$

With the transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{11}$$

and judicious choice of the elements of the transformation matrix, we get

$$\frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T^{-1}\begin{pmatrix} -1 + 2e_1\gamma & c - 2e_1\delta \\ -c + 2e_2\gamma & 1 - 2e_2\delta \end{pmatrix} T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Delta & \tau \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{12}$$

Therefore we now deal with the second-order damped Hill equation

$$\ddot{v}_1 - \tau\dot{v}_1 + \Delta v_1 = 0. \tag{13}$$

Utilizing the so-called Liouville transformation $v_1 = e^{4\int xydt} w$ we find the Hill's equation

$$w'' + \left(c^2 - 1 + 4c(1-c)\, y^2 - 4x^2\left(c^2 + c - 4y^2\right)\right) w = 0. \tag{14}$$

Substituting the x and y solutions this simplifies to

$$\begin{aligned} &w'' + f(t)\, w = 0, \\ &f(t) = \frac{(c-1)^2(c+1)((c-1)\left(1-4x_0^2\right) - 8x_0^4 + 4x_0^2(2x_0^2-1)\cos(2\Omega t))}{(c-1+2x_0^2(1-\cos(2\Omega t)))^2}. \end{aligned} \tag{15}$$

The stable and unstable regions are separated by the curve $|Tr(M)| = 2$, where M is the monodromy matrix generated by two independent solutions of the Hill's equation. The integral of $f(t)$ is evaluated as

$$\int_0^T f(t)\, dt = \frac{\sqrt{c-1}(c+1)(c - 1 - 2(2c-1)x_0^2)\pi}{\sqrt{c - 1 + 4x_0^2}\Omega}. \tag{16}$$

This integral is zero when $c = \frac{1-2x_0^2}{1-4x_0^2}$. Interestingly, this curve is very close to the numerically generated stability curve and captures the asymptotic behavior.

References

[1] Kerswell R. R. (2002) Elliptical instability. *Annual review of fluid mechanics* **34**(1):83-113

[2] Magnus W. and Winkler S. (2013) Hill's equation