Hopf Bifurcation in a Nonlinear Mechanical Model of Human Balancing with Delayed PDA Control

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Summary. Acceleration feedback improves human balancing in the presence of reflex delay. The corresponding human balancing model is a neutral delay differential equation. With consideration the nonlinearity in the inverted pendulum for large angular oscillations, the conditions for the occurrence and the sense of the Hopf bifurcation in the neutral system are presented.

Introduction

Efforts in interpreting human balancing behaviour must take into consideration intrinsic time delays arisen in the neural system [1]. The delayed balance control signals depend on the human sensory inputs which may include acceleration information. To take into account the acceleration signals, the simplest linear feedback controller must be a proportional-derivative-acceleration (PDA) controller [2]. It is shown in [3] that acceleration feedback improves human balancing against reflex delay.
There is an obvious nonlinearity in the mechanical part of the human balancing model, which is related to large angular oscillations of an inverted pendulum. The nonlinear analysis of the corresponding mechanical model requires an investigation on the resulting mathematical model that is a nonlinear neutral delay differential equation (NDDE). The rate of change of the state variables of an NDDE relies on the rate of change of the state values at earlier instances. Hence, an NDDE usually involves increasing complexities in an infinite phase space. In [4], the authors proposed a symbolic computation scheme for the purpose of Hopf bifurcation analysis for NDDEs. With the help of the symbolic computation methods, this study aims at a Hopf bifurcation analysis in the nonlinear mechanical model of human balancing with delayed PDA control when the reflex delay is considered as the bifurcation parameter.

Mechanical model

The inverted pendulum model of human balancing is shown in Figure 1. The human body is modelled as a rod of mass m, while l stands for the distance between the center of gravity C and the suspension point A. J_A is the moment of inertia with respect to the pin at A. The torsional spring of stiffness k_t models the forces in the human ankle. The control effort is the torque Q applied at the ankle additionally since the torsional spring is not strong enough to counterbalance the moment of the gravitational force. The single generalized coordinate is the angle φ of the human body. The equation of motion assumes the form

$$\ddot{\varphi}(t) - a\varphi(t) + c\varphi^3(t) = -k_p\varphi(t-\tau) - k_d\dot{\varphi}(t-\tau) - k_a\ddot{\varphi}(t-\tau) \tag{1}$$

where $a = (mgl - k_t)/J_A$ and $c = mgl/6J_A$ are originated in the third order approximation of the nonlinearity, τ is the feedback delay, $k_p = K_p/J_A$, $k_d = K_d/J_A$, $k_a = K_a/J_A$, where $K_{p,d,a}$ are the proportional, derivative and acceleration gains, respectively.

Linear analysis

By linear analysis, one can find the critical values of time delay τ_c in the form

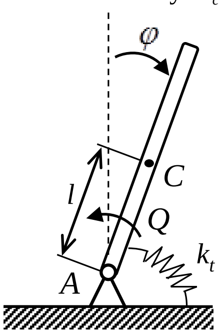

Figure 1 Mechanical model of postural balancing

$$\tau_{ck} = \frac{1}{\omega}\left[2k\pi + \arccos\left(-\frac{\omega^4 k_a + \left(ak_a - k_p\right)\omega^2 - ak_p}{\omega^4 {k_a}^2 + \left({k_d}^2 - 2k_a k_p\right)\omega^2 + {k_p}^2}\right)\right], \quad k = 0,1,2,\cdots \tag{2}$$

where $\omega = \sqrt{\dfrac{-p+\sqrt{p^2-4q(1-k_a^2)}}{2(1-k_a^2)}}$, $p = 2k_a k_p - {k_d}^2 + 2a$, $q = a^2 - {k_p}^2$.

Figure 2 demonstrates the stability region of the equilibrium $\varphi = 0$ in the plane (k_a, τ) with a realistic set of parameters [5].

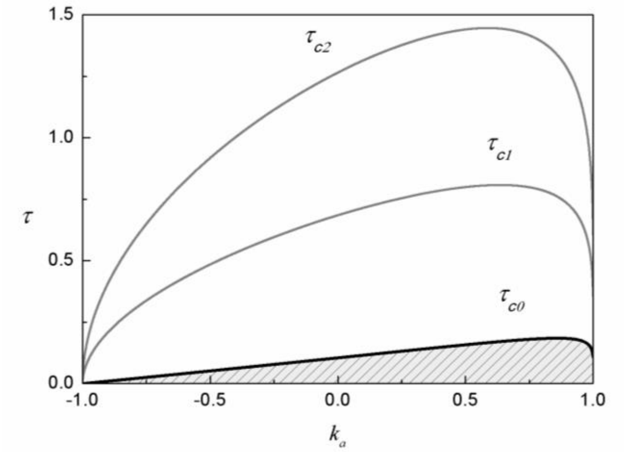

Figure 2 Stability diagram with $m = 60\,\text{kg}$, $l = 1\,\text{m}$, $J_A = 60\,\text{kg}\cdot\text{m}^2$, $k_t = 471\,\text{Nm/rad}$, $k_p = 50\,\text{s}^{-2}$, $k_d = 10\,\text{s}^{-1}$, where the shaded region ensures stable equilibrium.

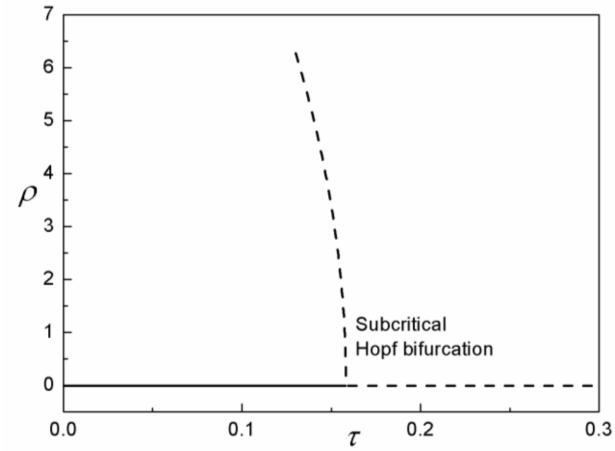

Figure 3 Bifurcation diagram with respect to τ with acceleration gain $k_a = 0.5$.

Hopf bifurcation

The symbolic code [4] results the normal form:

$$\begin{cases} \dot{y}_1 = i\omega_c y_1 + y_1\delta + \Delta_2 {y_1}^2 y_2 \\ \dot{y}_2 = -i\omega_c y_2 + y_2\delta + \bar{\Delta}_2 y_1 {y_2}^2 \end{cases} \tag{3}$$

where $\Delta_1 = w_2 \mathrm{e}^{i\omega_c \xi}\left(\left(-i\omega_c k_d - 2k_p\tau_c\right)\mathrm{e}^{-i\omega_c} - ib\omega_c + 2\tau_c a\right)$, $\Delta_2 = -3w_2{\tau_c}^2 c\mathrm{e}^{i\omega_c \xi}$,

$$w_1 = -\frac{\left(i\omega_c k_a + k_d\tau_c\right)\mathrm{e}^{-i\omega_c} + i\omega_c}{\left(i\omega_c k_d\tau_c - 2i\omega_c k_a - \omega_c^2 k_a + k_p{\tau_c}^2 - k_d\tau_c\right)\mathrm{e}^{-i\omega_c} - 2i\omega_c}, \quad w_2 = \frac{\mathrm{e}^{-i\omega_c \xi}}{(ik_a\omega_c + \tau_c k_d)\mathrm{e}^{-i\omega_c} + i\omega_c} w_1 .$$

The sense of Hopf bifurcation is determined by $\operatorname{sgn}\operatorname{Re}(\Delta_2) = \operatorname{sgn}((\omega_c^2 + a\tau_c^2)^2 - k_d\tau_c(\omega_c^2 k_a + k_p\tau_c^2))$, while the root tendency is determined by $\operatorname{Re}(\Delta_1) > 0$. Figure 3 illustrates the corresponding subcritical Hopf bifurcation.

Conclusion

To understand the nonlinear dynamics of the inverted pendulum model of human balancing with a delayed PDA controller, the conditions for the occurance of Hopf bifurcation and its sense which is subcritical for most of the realistic values are presented. This means that the linear PDA controller is not able to stabilize the vertical position for perturbations larger than the amplitude of the unstable limit cycle even when the delay is less than its critical value.

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