Set Oriented Numerical Methods for Spatially Dependent Parameter Uncertainty

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Summary. We propose a set oriented numerical approach which allows to quantify parameter uncertainty in dynamical systems from a global point of view. In this context we extend existing techniques which have been developed for the treatment of *static* parameter uncertainty to the situation where the underlying parameter uncertainty is *spatially dependent*. The efficiency of our new algorithms is illustrated by several numerical examples.

Introduction

Whenever one wants to analyze real-world dynamical phenomena by mathematical models, uncertainty with respect to both initial conditions and parameter dependencies is unavoidable. Often one has to quantify to what extent parameter uncertainties influence the numerical result of some computation in order to evaluate its significance. Obviously this type of *uncertainty quantification* (UQ) has a wide range of applications both in science and engineering.
For this reason, numerical techniques for uncertainty propagation in dynamical systems play an important role. In particular, *set oriented numerical methods* in combination with *transfer operator techniques* often prove to be useful for obtaining a statistical understanding of the dynamical behavior [2]. Recently, a set oriented numerical methodology has been developed [3] that allows to perform uncertainty quantification for autonomous dynamical systems from a global perspective. This methodology includes the approximation of global attractors, so-called (Q, Λ)-*attractors*, for dynamical systems with parameter uncertainty as well as the computation of corresponding invariant measures. More precisely, these attractors are lying within Q (in state space), and the parameter uncertainty is supported on $\Lambda \subset \mathbb{R}^p$.
In this article we extend existing techniques which have been developed for the treatment of *static* parameter uncertainty to the situation where the underlying parameter uncertainty is spatially dependent. In the following section we briefly summarize the set oriented approach. Then we explain how to extend this technology to the state dependent situation. The related algorithms have been integrated into the software package GAIO [1], and we finally illustrate the use of this new technique by a numerical example.

Computation of (Q, Λ)-Attractors via Subdivision

We consider parameter dependent discrete dynamical systems of the form

$$x_{j+1} = f(x_j, \lambda), \quad j = 0, 1, \ldots. \tag{1}$$

Here $x_j \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^p$ is the compact set of admissible parameter values. In order to take the uncertainty in (1) with respect to λ into account, we consider the corresponding (set-valued) map $F_\Lambda : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, where

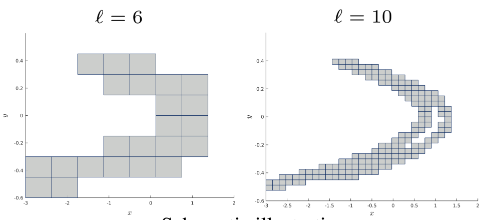
$$F_\Lambda(x) = f(x, \Lambda)$$

and $\mathcal{P}(\mathbb{R}^n)$ denotes the power set of $\mathbb{R}^n$. Then the (Q, Λ)-*attractor* is given by

$$A_{Q,\Lambda} = \bigcap_{j \geq 0} F_\Lambda^j(Q). \tag{2}$$

$A_{Q,\Lambda}$ can be viewed as the set which contains all the backward invariant sets of F_Λ. In particular, $A_{Q,\Lambda}$ contains all invariant sets of (1) corresponding to particular probability distributions on Λ.
For the computation of $A_{Q,\Lambda}$ we use the following subdivision scheme. Given an initial collection $\mathcal{C}_0$ of boxes $C \subset Q$ with $\bigcup_{C \in \mathcal{C}_0} C = Q$, we obtain $\mathcal{C}_\ell$ from $\mathcal{C}_{\ell-1}$ for $\ell = 1, 2, \ldots$ in two steps:

1. *Subdivision:* Construct a refined collection $\hat{\mathcal{C}}_\ell$ covering the same area,
$$\bigcup_{C \in \hat{\mathcal{C}}_\ell} C = \bigcup_{C \in \mathcal{C}_{\ell-1}} C \quad \text{and} \quad \operatorname{diam}(\hat{\mathcal{C}}_\ell) < \operatorname{diam}(\mathcal{C}_{\ell-1})$$
2. *Selection:* Define the new collection $\mathcal{C}_\ell$ by
$$\mathcal{C}_\ell = \left\{ C \in \hat{\mathcal{C}}_\ell : \exists \hat{C} \in \hat{\mathcal{C}}_\ell \text{ and } \exists \lambda \in \Lambda \text{ s.t. } f(\cdot, \lambda)^{-1}(C) \cap \hat{C} \neq \varnothing \right\}$$



Schematic illustration

If we denote by Q_ℓ the area in state space covered by $\mathcal{C}_\ell$, that is $Q_\ell = \bigcup_{C \in \mathcal{C}_\ell} C$, then obviously $Q_\ell \subset Q_{\ell-1}$. Moreover one can show that this nested sequence converges to $A_{Q,\Lambda}$ (see [3]):

$$\lim_{\ell \to \infty} Q_\ell = A_{Q,\Lambda}.$$

Once $A_{Q,\Lambda}$ is known the invariant measure corresponding to a specific choice of parameter uncertainty is computed via a *transfer operator* approach [2]. More precisely, based on the coverings $\mathcal{C}_\ell$ of $A_{Q,\Lambda}$ we use *Ulam's method* to construct a corresponding Markov chain and compute its stationary distribution as an approximation of the invariant measure.

Extension to Spatially Dependent Parameter Uncertainty

We now explain how to extend the set oriented UQ-technology from the previous section to the situation where the parameter uncertainty depends on the location in state space. In this case the parameter space $\Lambda = \Lambda(x)$ will in general depend on x and we have to compute (cf. (2))

$$A_{Q,\Lambda(x)} = \bigcap_{j \geq 0} F^j_{\Lambda(x)}(Q), \tag{3}$$

where now $F_{\Lambda(x)} = f(x, \Lambda(x))$. Using an adapted subdivision scheme we can again prove convergence and thus, we can approximate both $A_{Q,\Lambda(x)}$ and invariant measures corresponding to particular types of parameter uncertainty in a reliable way. Let us illustrate this by the following example: Consider the time-T-map $f(x, \lambda) = \varphi^T(x, \lambda)$ (φ the flow; $x = (x_1, x_2)$) of the *van der Pol system*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \lambda(1 - x_1^2)x_2 - x_1. \end{aligned} \tag{4}$$

In our computation we approximate $A_{Q,\Lambda(x)}$ for $Q = [-3, 3] \times [-4, 4]$, and we choose $\Lambda(x) = \Lambda(x_1)$ to be intervals linearly interpolated between $\Lambda(-3) = [0.95, 1.05]$ and $\Lambda(3) = [0.05, 1.95]$ (e.g. $\Lambda(0) = [0.5, 1.5]$). For each $\lambda \in \Lambda(x_1)$, $x_1 \in [-3, 3]$, the system possesses a stable periodic solution as well as an unstable equilibrium at the origin. In Figure 1 (a), (b) we show coverings of $A_{Q,\Lambda(x)}$ by choosing the time step $T = 2$.
Now we assume that for $\lambda \in \Lambda(x)$ the parameter uncertainty is given by a Gaussian $\lambda \sim \mathcal{N}(1, \sigma(x)^2)$ (truncated on $\Lambda(x)$ and normalized). For $x_1 \in [-3, 3]$ we choose the standard variation $\sigma(x) = \sigma(x_1)$ to be linearly interpolated between $\sigma(-3) = 0.02$ and $\sigma(3) = 0.4$. Using a transfer operator approach based on the box covering of $A_{Q,\Lambda(x)}$ – as briefly described at the end of the previous section – we can approximate the corresponding invariant measure. In Figure 1 (c) we show this approximation for the time step $T = 2$ and subdivision depth $\ell = 18$.
Finally observe that due to our choice of $\Lambda(x)$ and $\mathcal{N}(1, \sigma(x)^2)$ the approximations in Figure 1 are not invariant under the rotation by 180 degrees – as it has to be expected for the deterministic van der Pol oscillator. Rather the object spreads out more on its left and lower left parts. However, qualitatively this makes perfectly sense since the size of the interval $\Lambda(x_1)$ is increasing with x_1, we have chosen $T = 2$, and the period of the limit cycle is approximately 6.6 for fixed $\lambda = 1$.

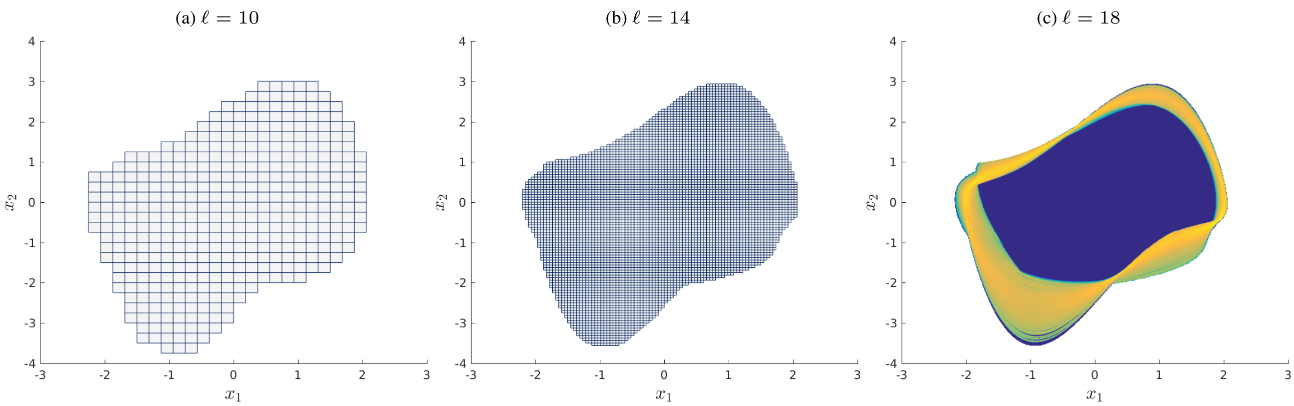


Figure 1: (a), (b) Box coverings of the $(Q, \Lambda(x))$-attractor $A_{Q,\Lambda(x)}$ of the van der Pol system for $T = 2$ obtained by the subdivision scheme after ℓ subdivision steps; $Q = [-3, 3] \times [-4, 4]$. (c) Invariant measure on $A_{Q,\Lambda(x)}$ for the parameter uncertainty $\lambda \sim \mathcal{N}(1, \sigma(x)^2)$. The density ranges from high density (yellow) to low density (dark blue).

Conclusions

Set oriented numerical techniques yield powerful tools for the analysis of parameter uncertainty in dynamical systems. Their advantage is that one obtains *global* insight by approximating the dynamical behavior in large parts of state space without relying on long term simulations of single trajectories. On the other hand the applicability of these techniques is currently restricted to problems with low dimensional state space.

References

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