Galerkin approximations for the Pole placement of time delayed systems

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Summary. In this work, an approach for designing feedback gains to stabilize the closed loop control systems having time delays is proposed. A new pseudo inverse method combined with Galerkin approximations is developed to obtain the characteristic roots of the system. Later, an optimization approach is used to find suitable feedback gains that stabilize the system. These gains ensure that the rightmost eigenvalue lies in the left half of the complex plane.

Introduction and mathematical modelling

In closed loop control systems, the communication lags between sensing and actuation can cause time delays to enter into the closed loop. These time delays can sometime destabilize the control system. In this work, the problem of stabilizing a time delayed control system is addressed [1, 2]. For a given time delay, a method is proposed for the selection of feedback gains, such that the closed loop system becomes stable. The dynamics of these time delayed systems are governed by DDEs, therefore their characteristic equations are quasi-polynomials and admit infinite roots. If all characteristic roots lies on the left half of the complex plane, the system is stable [3]. In order to design a method for stabilizing a time delayed system, one should know the information of the characteristic roots of the system. In this work the problem of finding the characteristic roots of the quasi polynomial is converted to a large linear eigenvalue problem. This is achieved by posing the DDE as an abstract Cauchy problem [4, 5], which is a partial differential equation (PDE) with time dependent boundary condition. This PDE is converted into a system of ordinary differential equations (ODEs) using Galerkin approximation. The boundary condition of the PDE is incorporated into the Galerkin approximation using a novel pseudo inverse method. As we increase the number of approximating ODEs for a given DDE, the right most eigenvalue of the system matrix of ODEs approaches the right most characteristic root of the given DDE. The following system of DDEs are considered:

$$\dot{\mathbf{x}}(\mathbf{t}) + \mathbf{A}\mathbf{x}(\mathbf{t}) + \sum_{q=1}^{m} \mathbf{B}\mathbf{K}_q^T \mathbf{x}(t-\tau_q) = 0, \quad \tau_q > 0 \tag{1}$$

Here, $\mathbf{x}(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_P(s)]^T$ are the states, $\mathbf{A}$ is a square matrix, $\mathbf{B}$ and $\mathbf{K}_q$ are column matrices. Given $\mathbf{A}$, $\mathbf{B}$ and τ suitable $\mathbf{K}_q$ needs to be found to stabilize the system. The optimal gains $\mathbf{K}_q^*$ are found by solving the following optimization problem:

$$\mathbf{K}_q^* = \text{arg min}(\text{Re}(\lambda_{max.}(\mathbf{K}_q)) + \alpha)^2, \qquad \alpha > 0 \tag{2}$$

Here, λ_{max} is the rightmost eigenvalue and α is the desired distance of the real part of λ_{max} from the imaginary axis in the left half of the complex plane. By introducing the transformation $\mathbf{y}(s,t) = \mathbf{x}(t+s)$ and by defining $\tau = \max\,[\tau_1, \tau_2, ..., \tau_q]$, Eq. (1) can be converted into an equivalent abstract Cauchy problem as shown below:

$$\frac{\partial \mathbf{y}}{\partial s} = \frac{\partial \mathbf{y}}{\partial t}, \quad -\tau \leq s \leq 0 \tag{3}$$

Boundary conditions for Eq. (3) are obtained by using $\mathbf{y}(0,t) = \mathbf{x}(t)$ and $\mathbf{y}(-\tau,t) = \mathbf{x}(t-\tau)$ and are given by:

$$\frac{\partial \mathbf{y}(0,t)}{\partial t} + \mathbf{A}\mathbf{y}(0,t) + \sum_{q=1}^{m} \mathbf{B}\mathbf{K}_q^T \mathbf{y}(-\tau_q, t) = 0 \tag{4}$$

The solution of Eq. (3) is assumed to be:

$$y_i(s,t) = \sum_{j=1}^{N} \phi_{ij}(s)\eta_{ij}(t); \quad i = 1,2,...,P; \quad j = 1,2,...,N \tag{5}$$

where $\phi_{ij}(s)$ are the basis functions and the $\eta_{ij}(t)$ are the time dependent coordinates. In vector form, Eq. (5) can be written as $\mathbf{y} = [\boldsymbol{\phi}_1(s)^T\boldsymbol{\eta}_1(t) \quad \boldsymbol{\phi}_2(s)^T\boldsymbol{\eta}_2(t) \quad \cdots \quad \boldsymbol{\phi}_P(s)^T\boldsymbol{\eta}_P(t)]^T = \boldsymbol{\Psi}(s)\boldsymbol{\beta}(t)$. Substituting Eq. (5) in Eq. (3), pre-multiplying with $\boldsymbol{\Psi}^T(s)$ and integrating over the domain $s \in [-\tau, 0]$, we get:

$$\mathbf{C}\dot{\boldsymbol{\beta}}(t) = \mathbf{D}\boldsymbol{\beta}(t) \tag{6}$$

where $\mathbf{C}$ and $\mathbf{D}$ are sparse rectangular matrices with $P \times NP$ dimensions. $\mathbf{C}^i = \int_{-\tau}^{0} \boldsymbol{\phi}_i(s)\boldsymbol{\phi}_i(s)^T ds$ and $\mathbf{D}^i = \int_{-\tau}^{0} \boldsymbol{\phi}_i(s)\boldsymbol{\phi}_i'(s)^T ds$ matrices are diagonal entries of $\mathbf{C}$ and $\mathbf{D}$. The symbol $'$ denotes the derivative with respect to s. Substituting Eq. (4) in Eq. (3) gives the boundary conditions to be incorporated in Eq. (6):

$$\bar{\mathbf{c}}\dot{\boldsymbol{\beta}}(t) = \bar{\mathbf{d}}\boldsymbol{\beta}(t) \tag{7}$$

where, $\bar{\mathbf{c}} = \big[\mathbf{\Psi}^T(0)\big]$ and $\bar{\mathbf{d}} = \big[-\mathbf{A}\mathbf{\Psi}^T(0) - \sum_{q=1}^{m} \mathbf{B}\mathbf{K}_q^T \mathbf{\Psi}^T(-\tau_q)\big]$. Equations (6) and (7) represent an over determined system of $P \times (N+1)$ equations having $P \times N$ unknowns in $\dot{\boldsymbol{\beta}}(t)$:

$$\mathbf{M}\dot{\boldsymbol{\beta}}(t) = \mathbf{K}\boldsymbol{\beta}(t), \quad \text{where} \quad \mathbf{M} = \begin{bmatrix}\mathbf{C}^T & \bar{\mathbf{c}}^T\end{bmatrix}^T \quad \text{and} \quad \mathbf{K} = \begin{bmatrix}\mathbf{D}^T & \bar{\mathbf{d}}^T\end{bmatrix}^T \tag{8}$$

The least-squares solution of Eq. (8) is given by $\dot{\boldsymbol{\beta}}(t) = \mathbf{M}^{+}\mathbf{K}\boldsymbol{\beta}(t)$, where, $\mathbf{M}^{+}$ is the Moore-Penrose inverse for the rectangular matrix $\mathbf{M}$. By defining $\mathbf{G} = \mathbf{M}^{+}\mathbf{K}$, Eq. (8) can be written as $\dot{\boldsymbol{\beta}}(t) = \mathbf{G}\boldsymbol{\beta}(t)$. The system of ODEs described by Eq. (8) approximate the system of DDEs shown in Eq. (1). The eigenvalues of $\mathbf{G}$ converge to the characteristic roots of Eq. (1) as the number of terms in the Galerkin approximation are increased [4].

Results and conclusions

Consider a third-order continuous time system represented in state-space model as follows [1]

$$\dot{\mathbf{x}} = \begin{bmatrix}-0.08 & -0.03 & 0.2\\ 0.2 & -0.04 & -0.005\\ -0.06 & 0.2 & -0.07\end{bmatrix}\mathbf{x} + \begin{bmatrix}-0.1\\ -0.2\\ 0.1\end{bmatrix}\mathbf{K}^T\mathbf{x}(t-\tau); \qquad \tau = 5 \tag{9}$$

By considering $N = 50$ in Eq. (5), we convert Eq. (9) into Eq. (8) and this results in $\mathbf{G}$ of size 150×150. Figure 1 shows the eigenvalues (shown in red dots) of $\mathbf{G}$. Also shown in blue dots are the eigenvalues of matrix G that have converged to the characteristic roots of Eq. (9). These converged eigenvalues (around 100) generate an absolute error of less than 1×10^{-4} when substituted into the characteristic equation of Eq. (9). The real part of the rightmost eigenvalue is 0.0232, implying that the system is unstable (max Re$(\lambda_{max}) > 0$). To stabilize Eq. (1), we minimize the objective function shown in Eq. (2) by selecting $\alpha = 1$. The optimization is performed using the Matlab function "$fminsearch$", which is based on the Nelder-Mead algorithm. The optimization process took 189 iterations to converge to a local minimum at which the value of the objective function was 0.8223. For the optimum gains $\mathbf{K}^* = [0.5473 \quad 0.8681 \quad 0.5998]^T$, the real part of the rightmost eigenvalue is located at -0.0932 and hence Eq. (9) is stabilized. Figure 2 shows the characteristic roots of Eq. (9) for optimal $\mathbf{K}^*$. Figure 3 shows the variation in the real part of the rightmost eigenvalues with respect to τ for the optimal $\mathbf{K}^*$. From the figure it is clear that for the optimal gains found, the system is stable for $\tau \in (0, 8]$. Using the

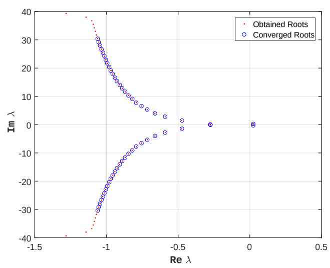

Figure 1: Characteristic roots of the system for initial guess $\mathbf{K}$

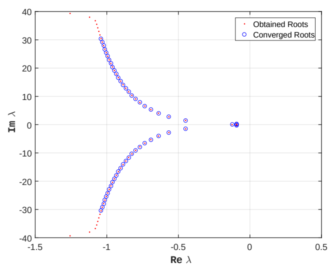

Figure 2: Characteristic roots of the system for optimal $\mathbf{K}$

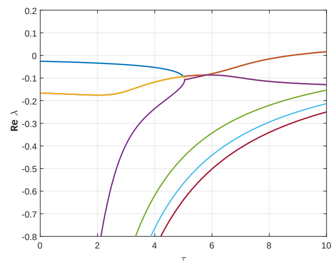

Figure 3: Variation of rightmost eigenvalues with respect to τ for optimal $\mathbf{K}$

current framework, we have obtained satisfactory results for the various test cases given in [1]. We have found that the pseudo inverse method is especially helpful in determining the stability of the DDEs using the ODE stability theory. It is also noticed that the pseudo inverse method is easier to implement on a computer compared to spectral-tau and spectral least-squares [4] and can easily be combined with an optimization framework.

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