Parameter estimation for nonsymmetric matrix Riccati differential equations

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Summary. Presented is an efficient method for estimating the parameters of nonsymmetric matrix Riccati differential equations (RDE) with constant coefficients using finite data. This method utilizes explicit inversion of the solution map and does not require numerical integration and optimization. It takes advantage of Radon's lemma which relates a solution of the RDE to a solution of a system of linear differential equations and a recently developed theory of assessing the uncertainty of solutions of linear dynamical systems.

Introduction

Inital value problems for symmetric and nonsymmetric Riccati differential equations (RDE) appear in many branches of applied mathematics, for example in variational theory, optimal control and filtering, dynamic programming and differential games. The most common occurrence is as an intermediate step in the determination of optimal control of a system in which the state equations are linear in the state and control variables [1], such as in the case of chemostat models [2]. RDE also constitute a special case of more general Lotka-Volterra systems that are commonly used as models of chemical and biological systems [3] and have been shown to be universal representatives of systems of ordinary differential equations with polynomial right-hand sides.

Theory

Consider systems described by the nonsymmetric matrix Riccati differential equation (RDE), i.e.,

$$\dot{X} = -XBX - XA + DX + C, \qquad X(0) = X_0 \tag{1}$$

where $X(t) \in \mathbb{R}^{m\times n}$ is the state of the system at time t, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$ are constant matrix parameters of the system, and X_0 is the initial condition. Let M and $Y(t)$ be defined as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad Y(t) = \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}$$

where $P(t), Q(t)$ are $m \times n$ and $n \times n$ matrix valued functions. Classical theory of Radon (see, e.g., [4]) states that any solution $X(t)$ of (1) is locally equivalent to a solution $Y(t)$ of the linear ODE system

$$\dot{Y} = MY \tag{2}$$

in the sense that

1. If $X(t)$ is a solution of (1) on some open interval $J \ni 0$ and $Q(t)$ is the unique solution of the initial value problem $\dot{Q} = (A + BX(t))Q,\ Q(0) = I$, then $Y(t)$ with $P(t) = X(t)Q(t)$ is a solution of (2) in the interval J with $Y(0) = (IX_0)^T$.
2. If $Y(t)$ is a solution of (2) on some open interval $J \ni 0$ such that $\det Q(t) \neq 0$ for $t \in J$ then $P(t)Q^{-1}(t)$ is a solution of (1) on the interval J with initial condition $X_0 = P(0)Q^{-1}(0)$.

Our goal is to determine the matrix coefficients A, B, C, D from a series of $p + 1$ observations $X_0, X_1, ..., X_p$ of the system made at equally spaced times. This is equivalent to solving a set of equations $X_k = X(kh; A, B, C, D)$ with $k = 1, 2, ..., p$, where $X(t; A, B, C, D, X_0)$ denotes the solution of the initial value problem (1) and h denotes the time step, and amounts to inverting the solution map that takes the parameters $\{A, B, C, D, X_0\}$ into the observations $\{X_0, X_1, ..., X_p\}$. Traditionally, inverse problems for nonlinear dynamical systems are solved by numerical minimization of the objective function

$$f(A, B, C, D) = \sum_{k=1}^{p} (X(kh; A, B, C, D, X_0) - X_k)^2$$

using appropriate optimization algorithms. Among the disadvantages of such methods are a potential poor convergence rate or convergence to a local optimum. In this work we describe a novel method that combines the equivalent linear formulation (2) and newly developed procedures for parameter identification of linear systems [5, 6].

Identification of parameters

Given a collection of observations $Y_0, Y_1, ..., Y_p$ of the linear system (2) one can easily determine the matrix of coefficients $M \in \mathbb{R}^{(m+n)\times(m+n)}$ as follows [6]: (i) Construct the matrices $\bar{Y}_0 = [Y_0|Y_1|...|Y_{p-1}]$ and $\bar{Y}_1 = [Y_1|Y_1|...|Y_p]$, (ii) Compute the matrix $\Phi = \bar{Y}_1\bar{Y}_0^{-1}$, and (iii) find M as the real matrix logarithm of Φ, i.e., a solution of the matrix equation $e^M = \Phi$. Results of Culver [7] define under what conditions such a logarithm exists and is unique.

Now, the knowledge of the data $X_0, X_1, ..., X_p$ does not immediately imply the knowledge of $Y_0, Y_1, ..., Y_p$, since for each k, X_k only determines the product $P_k Q_k^{-1}$, whereas $Y_k = (Q_k\ P_k)^T$. However, one can utilize the following observation: By definition,

$$\begin{pmatrix} Q_{k+1} \\ X_{k+1}Q_{k+1} \end{pmatrix} = Y_{k+1} = \Phi Y_k = \Phi \begin{pmatrix} Q_k \\ X_k Q_k \end{pmatrix}, \qquad k = 0, 1, ..., p-1 \tag{3}$$

where $Q_0 = I$. Assume that Q_k is invertible for $k = 0, 1, ..., p-1$. Then (3) reduces to a system of mnp linear equations for Φ:

$$\begin{pmatrix} X_{k+1} & -I \end{pmatrix} \Phi \begin{pmatrix} I \\ X_k \end{pmatrix} = 0, \qquad k = 0, 1, ..., p-1 \tag{4}$$

The system (4) is homogeneous and hence it specifies Φ only to within a scalar multiple. This corresponds to the fact that changing A and D to $A + \alpha I$ and $D + \alpha I$ does not alter the solution of (1). Identifiability can be recovered by restricting the set of permissible M to traceless matrices.

Example

Consider a model of a system with two interacting populations with constant influxes and growth rates that are limited equally by an environmental stress that depends linearly on the sizes of the populations. The resulting equations can be written as

$$\dot{x_1} = c_1 + g_1 x_1 - x_1(b_1 x_1 + b_2 x_2)$$
$$\dot{x_2} = c_2 + g_2 x_2 - x_2(b_1 x_1 + b_2 x_2)$$

This system can be rewritten as RDE of the form (1) with $m = 2$, $n = 1$, $X = (x_1\ x_2)^T$, $A = a$, $B = (b_1\ b_2)$, $C = (c_1\ c_2)^T$, and D a diagonal matrix with $d_{11} = g_1 + a$ and $d_{22} = g_2 + a$. Special case with $(b_1, b_2, c_1, c_2, g_1, g_2) = (0.5, 0.5, 1.5, 0.5, 0.5, 1.0)$ with initial condition $X_0 = (0\ 1)^T$ produces trajectory seen in Figure 1. The system is uniquely identified with data observed for 5 time points, as indicated.

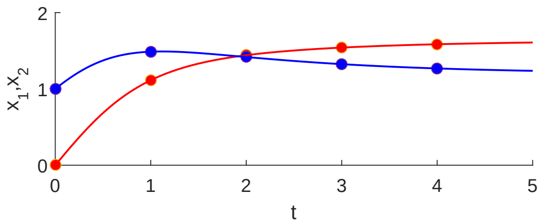

Figure 1: Trajectory of x_1 (red) and x_2 (blue) and calibration data for an example RDE system.

Conclusions

Presented is a procedure for estimation of parameters of nonsymmetric Riccati differential equations using exact data. This procedure takes advantage of the availability of explicit solutions of the system and avoids complications stemming from numerical estimation of the parameters for this nonlinear ODE system, such as slow convergence, false convergence to local optima, etc. Explicit nature of the solution allows for forward and backward error analysis of the problem which will be forthcoming.

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