Effect of Softening Constitutive Law on Column Buckling

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Summary. Columns with linear constitutive laws in bending can sustain higher than critical buckling load after the onset of buckling. In other words, they have stable post-buckling behavior. However, several materials exhibit cubic nonlinearity in the constitutive law with softening effect. An analysis with simple idealized model reveals that such cubic nonlinearity in the constitutive law diminishes the domain of post-buckling stability, and leads to catastrophic collapse. This work further investigates the finding by taking into account the nonlinear constitutive law into beam theory.

Introduction

A class of materials such as redwood exhibits softening non-linearity in the constitutive law [1, 2] of the following form:

$$\sigma = E_1\epsilon + E_2\epsilon^3, \tag{1}$$

where $E_1(>0)$ and $E_2(<0)$ are constants, σ is normal stress and ϵ is normal strain. For a beam under bending, this constitutive law results in the following restoring moment

$$M = E_1 I_1 \kappa + E_2 I_2 \kappa^3, \tag{2}$$

where κ is the curvature of the beam and I_1 and I_2 are the 2nd and 4th moments of area about the neutral axis, respectively. An analysis with simple idealized model (see Fig. 1) reveals that such cubic non-linearity in the constitutive law diminishes the domain of post-buckling stability, and can lead to catastrophic collapse.

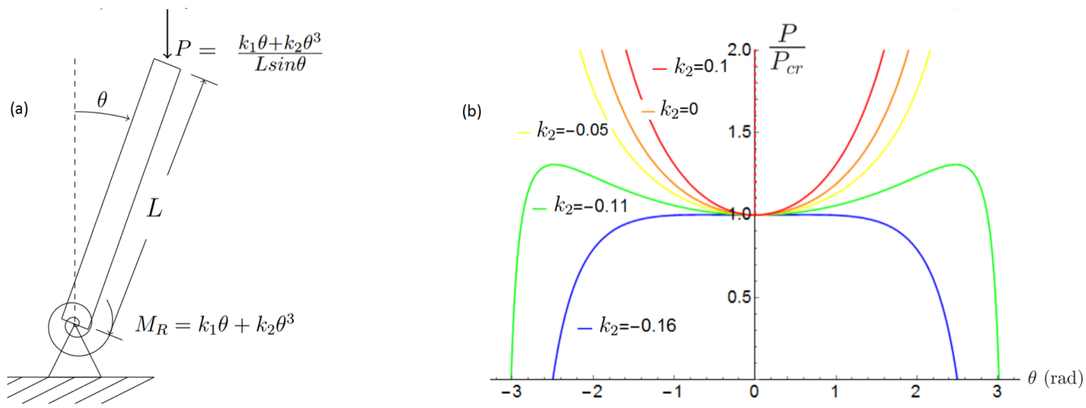

Figure 1: (a) Idealized model of buckling of a clamped-free column where $k_1 = 1Nm/rad$, $L = 1m$. (b) This buckling bifurcation diagram shows that as k_2 (Nm/rad^3) decreases, the stable buckling (red, orange and yellow curves) transitions through a partially stable (green curve) buckling to an unstable buckling (blue curve).

Inspired by this insight, we investigate the post-buckling stability of beams with softening constitutive laws using perturbation analysis. An overarching goal of this research is to analyze buckling of ‘microtubules’. Microtubules are cytoskeletal filaments, which provide structural support to the cell, and play an important role in the cell motility and cell division. Their bending deformation and buckling are involved directly in the myosin contractility [3]. Our coarse-grained molecular dynamics (MD) simulations reveal that microtubules have softening constitutive law in bending [4].

Perturbation Analysis of Beam Under Buckling

To find the relationship between load P and deflection b (see Fig. 2) in the post-buckling regime, we applied perturbation theory to the Euler equation of the buckled beam derived from the following potential energy function:

$$V_{total} = \int_0^L \frac{1}{2} M_R \kappa ds - P\Delta \tag{3}$$

where Δ is the displacement in the x-direction. The deflection function $w(s)$ and load P are expanded in terms of the perturbation parameter (the deflection b) as

$$w(s) = w_1(s)b + w_2(s)b^2 + w_3(s)b^3 + ... + w_n(s)b^n \tag{4}$$

$$P = P_{cr} + P^{(1)}b + P^{(2)}b^2 + P^{(3)}b^3 + ... + P^{(n)}b^n \tag{5}$$

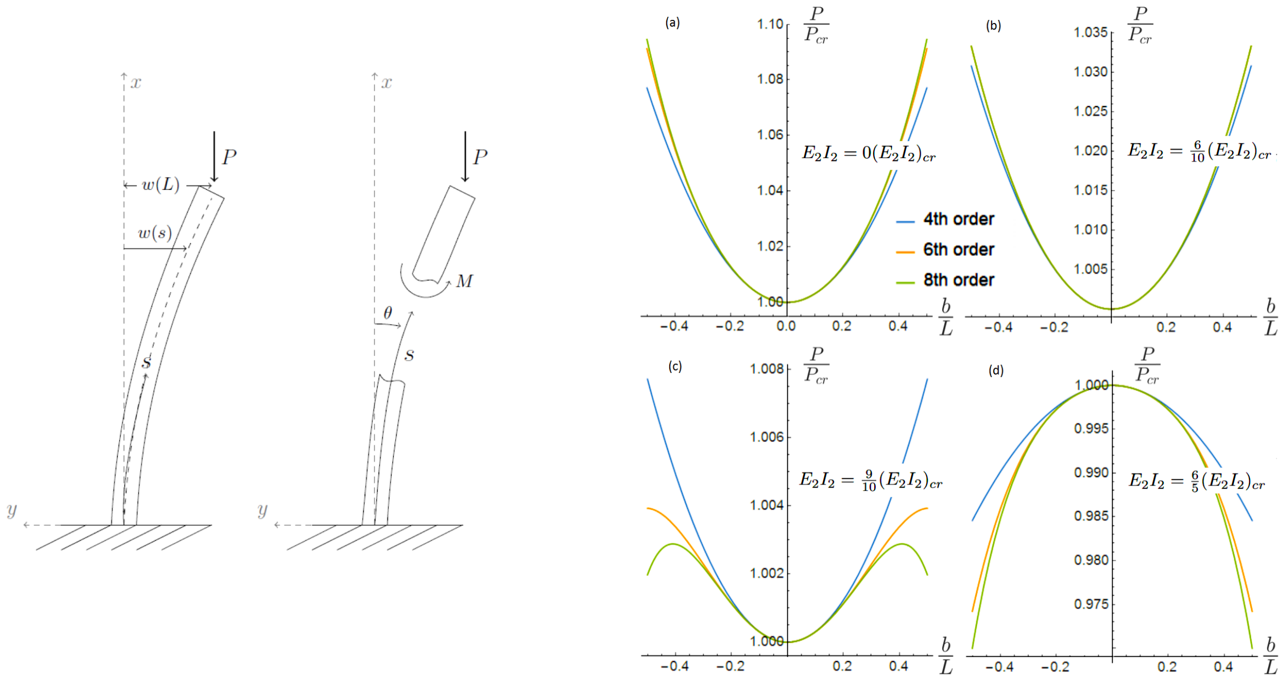

Figure 2: (Left) Column buckling (clamped-free). (Right) Load vs. deflection from perturbation analysis (Eq. 6) in increasing order of accuracy to show convergence. Similar to the idealized model (Fig. 1), the 6th and higher order solutions (orange and green curves) show that as E_2I_2 decreases from 0 to below $(E_2I_2)_{cr}$, the stable buckling (a) becomes unstable (d) transitioning through a partially stable buckling (c) for $0 > E_2I_2 > (E_2I_2)_{cr}$. However, the 4th order solution [2] (blue curve) shows an abrupt change from stable to unstable buckling at $(E_2I_2)_{cr}$

and substituted into the Euler equation. Then, we derived n-th differential equation for $w_n(s)$ and P_{n-1} based on the coefficients of b^n and finally solve for the load P as a function of deflection b:

$$P = \frac{\pi^2 E_1 I_1}{4L^2} + \frac{\left(2\pi^4 E_1 I_1 L^2 + 3\pi^6 E_2 I_2\right)}{256 L^6} b^2 + \frac{\pi^6\left(76(E_1I_1)^2L^4 + 148\pi^2 E_1 I_1 E_2 I_2 L^2 - 21\pi^4 (E_2I_2)^2\right)}{131072 E_1 I_1 L^{10}} b^4 + O(b^6) \tag{6}$$

The 4th oder accurate solution ($O(b^4)$) [2] fails to capture the partially stable buckling seen in the idealized model. We solved for the higher order solutions (with 6th, 8th and 10th order accuracies), all of which reveal the existence of partially stable buckling for $0 > E_2I_2 > (E_2I_2)_{cr}$, where $(E_2I_2)_{cr} = -\frac{1}{6}E_1I_1(\frac{2L}{\pi})^2$ (see Fig. 2).

Conclusions

This work analyzes the pos-buckling dynamics of beams exhibiting cubic nonlinearity in their constitutive laws with softening effect. In particular, we show how a stable post-buckling behavior changes to an unstable behavior (leading to catastrophic collapse under a fixed load) as the softening nonlinearity is increased. Starting with a linear constitutive law, as we increase the cubic factor contributing to the softening nonlinearity, we find that the domain of post-buckling stability diminishes finally leading to the unstable behavior. This work has relevance to engineering structures as well as biological filaments, for which buckling dynamics are crucial.

References

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